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A COMMENT ON DOREIAN'S REGULAR EQUIVALENCE IN SYMMETRIC STRUCTURES

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I read with interest Pat Doreian's recent article (1987) on the problem of running REGE (White 1984) on symmetric matrices. As many people have discovered to their chagrin, the problem is that in a symmetric matrix, REGE finds that all actors except isolates are equivalent to all others no matter what the data. This is troubling in cases where clear differences in structural position appear to exist between some actors. Whereas for directed graphs the results of REGE correspond closely with intuitive notions of role (Nadel 1957; Sailer 1978; Faust 1985), for symmetric data this correspondence seems to break down. Doreian's solution, which I call the "Doreian Split", is creative and practical, and yields intuitively satisfying results.

If there really is a problem, that is. Doreian illustrates the situation with the graph shown in Figure 1. Most people immediately spot four groups of equivalent actors: (a), (b, c), $(d,\cdot\cdot g)$, and $(h\cdot\cdot o)$. Actors within each group are similarly related to members of other groups, a condition consistent with many intuitive notions of role, and with the notion of regular equivalence (White and Reitz 1983) in particular. Doreian explains it this way:

> In this structure there are 4 equivalence classes: (i) nodes h through o are equivalent; (ii) nodes d through g are equivalent; (iii) nodes b and c are equivalent; (iv) and a is equivalent only to itself. Nodes h and i are connected to d in the same way that nodes l and m are connected to f and, finally, nodes n and o are connected to g in the same way. It is clear also that nodes d and e are connected in the

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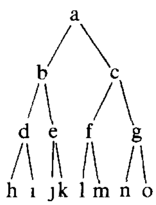

Figure 1

> same way to b as nodes f and g are connected to c. Finally, b and c are identically linked to s. *The definition of regular equivalence gives exactly this set of equivalence classes. However, when REGE is used, all nodes are returned as equivalent to each other.* (Doreian 1987: 91, italics added)

The situation, then, is that the intuitively satisfying four-group partition is consistent with the algebraic definition of regular equivalence, but the operational implementation (REGE) gets it wrong.

Or does it? The relevant White and Reitz theorems and definitions are as follows:

> *Definition 6.* A full graph homomorphism $f: G \to G'$ is regular if and only if for all a, b in P,
>
> $f(a)R'f(b)$ implies there exist c, d in P such that
>
> cRb, aRd, $f(c)=f(a)$ and $f(d)=f(b)$
>
> (White and Reitz 1983: 197)
>
> *Definition 11.* If $G=\langle P, R\rangle$ and $\equiv$ is an equivalence relation on P then $\equiv$ is a regular equivalence if and only if for all a, b, c, in P, $a \equiv b$ implies
>
> (i) aRc implies there exists $d \in P$ such that bRd and $d \equiv c$; and
>
> (ii) cRa implies there exists $d \in P$ such that dRb and $d \equiv c$.
>
> (White and Reitz 1983: 200)
>
> *Theorem 2C.* The equivalence induced by a regular graph homomorphism is a regular equivalence relaion and conversely every regular equivalence relation is induced by some regular homomorphism. (White and Reitz 1983: 201)

Now consider the following partition on the graph in Figure 1:

BLOCKMODEL A:

(a) (b) (c) (d) (e) (f) (g) (h, i) (j, k) (l, m) (n, o)

Is this partition consistent with the notion of regular equivalence? Yes. Blockmodel A represents structural equivalence (in the sense of Lorrain and White 1971), and White and Reitz prove (1983: 199) that all structural equivalences are also regular equivalences.

Consider also this partition,

BLOCKMODEL B:
$(a)\ (b)\ (c)\ (d)\ (e)\ (f)\ (g)\ (h)\ (i)\ (j)\ (k)\ (l)\ (m)\ (n)\ (o)$

and more importantly, this one:

BLOCKMODEL C:
$(a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i,\ j,\ k,\ l,\ m,\ n,\ o)$

All three blockmodels, including the one where all points are equivalent to all others, are consistent with regular equivalence. Let us consider why this is so. In plain English, the definitions and theorems above amount to a description of groups or blocks of equivalent actors such that if there is a directed tie between a member of one block with a member of another block, then at the block level there is a corresponding tie between the two blocks. Furthermore, if such a tie exists at the block level, then every member of each block will exhibit a corresponding tie, identically directed, with one or more members of the other block. In even plainer English, if it can be said that block A is doing it to block B, then every member of A is doing it to some member of B, and every member of B is having it done to by someone in A.

Let's check that blockmodel C is really consistent with regular equivalence. It maps all points to just one block (call it x) with a single bi-directional tie to itself (R': $x \leftrightarrow x$). Is everyone in x doing it to someone in x? Yes. Is everyone in x receiving it from someone in x? Yes. Therefore, we have a regular equivalence.

The same will be true of the other partitions noted above, plus a few others not shown. So what does this mean? Clearly, a graph may have several regular equivalences, nested hierarchically within each other, ranging from the overly exclusive (each actor equivalent only to himself), to the overly inclusive (all actors equivalent to all others). However, White and Reitz show that

> *Theorem 3C.* The collection of all regular equivalence relations on a graph has a maximal element.
>
> (White and Reitz 1983: 199)

In the case of the graph in Figure 1, the maximal regular equivalence relation is the one that maps all points to a single block. The REGE algorithm, designed to find the maximal equivalence relation, is not technically in error here. The problem is not in the algorithm, but in the definition. Now let us consider Doreian's discoveries in this light. I can't honestly follow the mathematical argument, but I believe him when he concludes:

> It follows that the decomposition of a symmetric structure into two asymmetric structures, by the use of centrality scores as attributes of the nodes, preserves regular equivalences. (Doreian 1987: 97)

But it doesn't preserve all of them, for when he applies his method to the graph in Figure 1, it does not find the maximal regular equivalence. Instead, it finds another one. The question is, which one? For Figure 1, it finds a good one, the next-most-maximal-yet-not-trivial one. But does it always choose the next-most maximal one? What exactly does it do when there are no exact equivalences to be found? One of these questions is answered by the graph in Figure 2. One set of regular equivalences on this graph is as follows:

1. $(a)\,(b)\,(c)\,(d)\,(e)\,(f)$
2. $(a)\,(b)\,(c)\,(d,\,e)\,(f)$
3. $(a)\,(b,\,c)\,(d,\,e,\,f)$
4. $(a,\,b,\,c,\,d,\,e,\,f)$

Clearly, the best ones are #2 and #3, and, if one is persuaded by the notion that two actors may play the same role and yet not know the same number of people, #3 is the better of the two. Doreian's method, however, gives #2, which represents structural equivalence (and are the orbits of the graph as well). The reason is that the Doreian Split depends upon centrality (either betweenness- or closeness-based) and centrality is sensitive to degree (Freeman 1978). Therefore, the only regular equivalences that can be preserved by the Doreian Split are

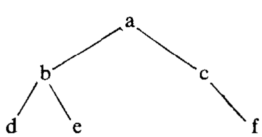

Figure 2

those that also group actors by degree. This may be a problem because researchers often turn to regular equivalence algorithms precisely because it is insensitive to degree: two judges are thought to play the same role even if they see different numbers of criminals (*cf.* Sailer 1978).

To use Doreian's method to detect degree-free regular equivalences, it is necessary to modify the measure of centrality. There are several ways to do it. For example, instead of betweenness centrality, one might compute what I call *generation centrality*. In a directed, perfectly hierarchical graph, generation centrality is a function of the number of levels above and below a given actor. In general, it is a function of the longest geodesic distance to and from an actor. For symmetric or undirected data, this reduces to a single number. If this version of centrality is used, Doreian's method gives equivalence #3 above, as desired. Numerous other distance and/or centrality measures work as well.

However, it is not always the case that one wants to ignore degree. Everett (1985) has argued that the equivalence arising from the *orbits* of a graph is useful representation of role. Everett describes this equivalence as follows:

> An automorphism of graph $G(V, R)$ is a permutation π of the vertices V which has the property that $(a, b) \in R$ if and only if $(\pi(a), \pi(b)) \in R$. Note that the set of all automorphisms of G form a group under the operation of composition which is denoted by $\mathrm{Aut}(G)$.
>
> Two vertices $a, b \in V$ belong to the same orbit of G if and only if $\pi(a) = b$ for $\pi \in \mathrm{Aut}(G)$.
>
> It is a simple exercise to verify that belonging to the same orbit is an equivalence relation.
>
> The above definitions can easily be extended to networks in which a variety of different relations are acting on the same vertex set.
>
> Obviously, any two vertices which are structurally equivalent will belong to the same orbit. Hence this new condition of role similarity is weaker than structural equivalence.
>
> It is stronger than regular equivalence as shown by the next theorem.
>
> *Theorem 1.* The equivalence relation of belonging to the same orbit of a graph $G(V, R)$ is a regular equivalence.

> *Proof.* Suppose a and b belong to the same orbit. Let $c \in V$ and suppose aRc. Since a and b belong to the same orbit, there exists a $\pi \in \mathrm{Aut}(G)$ such that $\pi(a) = b$. Hence $bR\pi(c)$ then we satisfy the first part of the condition for regular equivalence. The second part for cRa is similar.
>
> A consequence of this theorem is that our definition of role similarity differentiates between actors playing similar roles to different numbers of other actors. Hence if we consider the example of parents and children, structural equivalence would place together parents of the same children. Regular equivalence would place together all parents. The orbits would consist of all parents with an equal number of children. (Everett 1985: 355)

Orbits, then, are regular equivalences that preserve, among other things, degree. While it would be difficult to prove that the unmodified Doreian Split always finds the orbits of a graph, it is at least apparent that its results will be similar (and will always entail the orbits).

Summary

The Doreian Split implements a strengthening of regular equivalence that enables it to find meaningful structure in symmetric graphs. It is not, however, a new algorithm for finding maximal regular equivalences. Rather, it detects something intermediate between Lorrain and White structural equivalence and White and Reitz regular equivalence. The properties of the Doreian equivalence are consistent with those of "orbit" or "automorphic" equivalence, but we cannot be sure they are one and the same. In fact, we cannot even be sure that the Doreian Split always finds the same regular equivalence.

Perhaps what is really needed is to treat the detection of regular equivalence as a hierarchical clustering problem. Instead of blindly picking an unknown regular equivalence, as does the Doreian Split, and instead of arbitrarily choosing the maximal regular equivalence, as does REGE, we would prefer to see the whole tree of nested equivalences. Then we could choose the level of data reduction appropriate for the analysis at hand.

For asymmetric data, we would generally choose the maximal element, as REGE assumes, but not always: in some cases we would want to separate otherwise equivalent actors on the basis of degree, centrality, or some other network attribute. For example, in some applications it would not make sense for two actors to be perfectly equivalent, yet one be highly central while the other entirely peripheral. In such cases we would want to drop down to a less abstract equivalence that preserved some of the more basic graph measurements. Sometimes we would even want the next-to-*minimal* equivalence, which would normally be Lorrain and White's structural equivalence (Lorrain and White 1978; Burt 1983).

For symmetric data, we would always disregard the maximal regular equivalence, but would still be faced with a choice among the remaining equivalences.

And in graphs where no perfect equivalence existed, we must decide how to trade off the fit of a given partition against the amount of data reduction achieved.

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