

A NOTE ON JUNCTURE HOMOMORPHISMS *

Martin G. EVERETT **

Thames Polytechnic

Stephen P. BORGATTI †

University of South Carolina

We show that a juncture homomorphism does not imply that the associated semigroups of a network and its image are isomorphic.

1. Introduction

White and Reitz (1983) investigated the relationships between graph and semigroup homomorphisms on networks of relations. A graph has associated with it a semigroup in which the binary operation is relational composition. Simplification of the semigroup is usually achieved by looking for homomorphisms onto smaller simpler semigroup structures. The graph itself can be simplified by a graph homomorphism which gives rise to an image-graph which also has a semigroup associated with it. Obviously it would be desirable for the two semigroups (the original graph and the image graph) to be homomorphic for a given graph homomorphism. In other words, if a graph G has S as its associated semigroup and $f: G \rightarrow G'$ is a graph homomorphism then we would like S' , the semigroup associated with G' , to be homomorphic to S . This idea is summed up by the commutative diagram in Figure 1. White and Reitz (1983) prove that if f is a regular graph homomorphism then S and S' are homomorphic.

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** School of Mathematics, Statistics and Computing, Thames Polytechnic, Wellington Street, London SE18 6PF, U.K.

† Department of Sociology, University of South Carolina, Columbia, SC 29208, U.S.A.

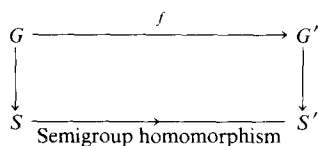


Fig. 1.

It is possible for a regular graph homomorphism to give a simplification of the graph and for the image to still have the same semigroup. In other words the operation of relational composition is unchanged under the graph homomorphism and the two semigroups S and S' are isomorphic. It should be noted that this situation is highly desirable but unfortunately it is often difficult to obtain much simplification of the data whilst simultaneously preserving the semigroup structure. White and Reitz (1983) prove that if f is strong then S and S' are isomorphic. However, they note that the stringent condition of being strong makes the concept far too restrictive for practical purposes.

All the above concepts can be extended to deal with networks of relations. (All formal definitions are contained in the appendix.) A network in which each pair of relations are disjoint is called a multiplex graph. Any network induces a multiplex graph via the bundle of relations between vertices. It is therefore desirable for the property of multiplexity to be preserved by any network homomorphism. Regular homomorphisms do not preserve multiplex graphs and therefore are not entirely satisfactory for modelling concepts of social role. The concept of preserving multiplexity can be related directly to local role equivalence in terms of a bundle homomorphism. White and Reitz (1983: 209) state:

To capture the global role structure of multiplex social relationships, we require a stronger homomorphism, one which has properties of both the bundle homomorphisms and the regular homomorphism. The strong network homomorphism has both properties, but is too restrictive.

It is interesting to note that what is asked of a network homomorphism is preservation of multiplexity together with semigraph homomorphism. The example of a strong homomorphism has the additional advantage of being a semigroup isomorphism.

White and Reitz propose the juncture homomorphism as a weaker homomorphism with both properties. They make the following claims.

...juncture homomorphisms have the desired properties of preserving multiplexity and preserving the semigroup of relations without the restrictiveness of a strong homomorphism. (pp. 209–210)

Juncture homomorphisms share with strong homomorphisms the property of preserving composition of relations. (p. 210)

These two statements, plus others in the paper, are claiming that juncture homomorphisms imply that the semigroup associated with a network is isomorphic to the semigroup associated with the image network. *This claim is false.* There is no isomorphism between the semigroups; however, since every juncture homomorphism is regular they are homomorphic. (We note this property was all that was initially required.) The following section contains a counter-example to the claim.

2. A counter-example

Figure 2 is a network of two relations R_1 and R_2 on 6 points. It is a simple matter to verify that the homomorphism $f_1(1) = f_1(4) = 1'$, $f_1(2) = f_1(5) = 2'$, $f_1(3) = f_1(6) = 3'$, $f_2(R_1) = R'_1$, $f_2(R_2) = R'_2$ is a juncture homomorphism. The image is given in Figure 3.

The semigroup associated with the network in Figure 2 has 30 elements. In contrast the semigroup associated with the image network

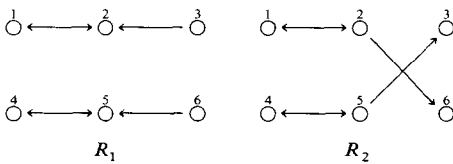


Fig. 2.

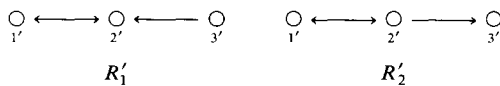


Fig. 3.

in Figure 3 has only 8 elements. The two semigroups cannot therefore be isomorphic. (However, since the juncture homomorphism is regular they are homomorphic.)

It is easy to see why White and Reitz believed the semigroups were isomorphic when you look at their theorem 13 (p. 210):

Theorem 13. Let $f: N \rightarrow N'$ be a juncture network homomorphism where $N = \langle P, \mathbb{R} \rangle$, $N' = \langle P', \mathbb{R}' \rangle$ and $f = \langle f_1, f_2 \rangle$. If o is relational composition and $\langle \mathbb{R}, o \rangle$ is a semigroup then $f_2: \langle \mathbb{R}, o \rangle \rightarrow \langle \mathbb{R}', o \rangle$ is an isomorphism.

The statement and proof of this theorem are both correct! However, the claims made in the text go further than the theorem. The hypothesis of the theorem assumes that $\langle \mathbb{R}, o \rangle$ is a semigroup. *This is not usually true.* We are interested in the semigroup generated by \mathbb{R} not in the small number of cases that $\langle \mathbb{R}, o \rangle$ is a semigroup. The network of Figure 2 requires graphs of an additional 28 relations to be both a network and a semigroup. It would be most unusual for a network to be a semigroup, and so in nearly all cases the theorem simply does not apply.

Appendix

We follow the notation of White and Reitz (1983). A network is a pair $N = \{ P, \mathbb{R} \}$ where P is a set of points and \mathbb{R} is a family of relations on P .

Let $N = (P, \mathbb{R})$ and $N' = (P', \mathbb{R}')$ be two networks. A full network homomorphism $f: N \rightarrow N'$ is an ordered pair of mappings (f_1, f_2) such that $f_1: P \rightarrow P'$ and $f_2: \mathbb{R} \rightarrow \mathbb{R}'$ are onto, for every $a, b \in P$ and $R \in \mathbb{R}$, $aRb \Rightarrow f_1(a)f_2(R)f_1(b)$, and for every $x, y \in P'$ and $R \in \mathbb{R}$, $xf_2(R)y \Rightarrow \exists c, d \in P$ such that $f_1(c) = x$, $f_1(d) = y$ and cRd .

A full network homomorphism $f: N \rightarrow N'$ is a regular network homomorphism if for each $R \in \mathbb{R}$ $f_1(a) f_2(R) f_1(b) \Rightarrow \exists c, d \in P$ such that $f_1(a) = f_1(c)$, $f_1(b) = f_1(d)$, cRb and aRd for all $a, b \in P$.

In a network N the bundle of relations B_{ab} from a to b where $a, b \in P$ is given by $B_{ab} = \{ R \in \mathbb{R} : aRb \}$.

If $f: N \rightarrow N'$ is a regular network homomorphism then f is a juncture homomorphism if and only if for all $a, b, c, d \in P$

$f_1(a) = f_1(c)$ and $f_1(b) = f_1(d)$ implies one of the following:

- (i) $B_{ab} = B_{cd}$
- (ii) $B_{ab} = \phi$
- (iii) $B_{cd} = \phi$

If N is a network then the semigroup S associated with N is the semigroup with \mathbb{R} as generators under relational composition.

References

- White, Douglas R. and Karl P. Reitz
1983 "Graph and semigroup homomorphisms on networks of relations". *Social Networks* 5:
193–235.