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A TESTING EXAMPLE FOR POSITIONAL ANALYSIS TECHNIQUES *

M.G. EVERETT **
Thames Polytechnic

Steve BORGATTI †
University of South Carolina

In trying to assess the validity of formal definitions of positional analysis it is necessary to have certain standard test data. The mathematical literature contains a large collection of important graphs specifically constructed as counter-examples to conjectures. We show that one of these graphs is useful in understanding the workings of certain positional analysis techniques.

1. Introduction

There are now a number of formal definitions together with practical algorithms which try and capture the concept of role in social networks. Recently articles have been appearing which try to examine the various different ideas by comparing their performance on real and hypothetical data (Faust 1988; Doreian 1988a, 1988b).

Practical algorithms are usually constructed by relaxing the formal conditions of the strict definition in some way. This allows a greater aggregation than would have been possible by simply applying the concepts directly in their rigid form. The relationship of the relaxed algorithmic formulation to the original definition is of course of prime importance in understanding any positional analysis of real data. However, it is of *fundamental importance* that the formal definition correctly captures our concept of positional analysis. To test whether this is the case we must test the definitions on artificially constructed

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** School of Mathematics, Statistics and Computing, Thames Polytechnic, London SE18 6PF, U.K.
† Department of Sociology, University of South Carolina, Columbia, SC 29208, U.S.A.

data where the role structure is obvious. In the articles cited above the authors have taken this approach and have also considered various methods on well-structured real data sets. Unfortunately it can be difficult to think of examples which are sufficiently demanding so as to provide insight into the performance of the various definitions. Commonly, researchers use a tree to represent some hierarchical organisation and their comparisons are based on how each method copes with this simple structure. Whilst it is true that any concept of positional analysis should be able to cope with a tree (failure in this case should be sufficiently damning to reject the concept) it should not be grounds for embracing the definition as successful. The thrust of positional analysis techniques should be highlighted by the term "position"; actors are aggregated because of where they are in the network, not because they share a particular attribute. If we examine the tree structure we see that a number of simple concepts will easily partition the nodes into realistic role sets. Suppose in our hierarchical structure we examine the relationship of "communication with". In a "perfect" organisation this should lead to an undirected tree which contains the same links as the formal organisational structure. In this case centrality, for example, will identify workers, middle managers, managing directors, etc. It follows that we must consider whether a particular definition is aggregating because of positional concepts or because of some attribute such as centrality. There is nothing wrong with using centrality or any other attribute, provided we realise explicitly that this is the case and it is position with respect to this attribute that is required.

Doreian (1987) suggests using centrality explicitly with REGE to find regular equivalences in symmetric graphs. He proposes splitting a symmetric relation into two asymmetric ones and submitting these to REGE. Each symmetric arc is replaced by a directed arc, directed from less central to more central in one relation and reversed in the other. The technique is fine as long as it is understood that the positional analysis is taking account of centrality. The result is simply the maximal regular equivalence that preserves centrality. There is no compelling reason why the attribute should be centrality. Other attributes such as degree of eccentricity are equally commendable. Borgatti and Everett (1988) propose a technique which separates relations and attributes and allows the researcher to find the maximal regular equivalence compatible with any attribute.

Centrality certainly plays an important role in social network analysis, including positional analysis. It is reasonable to ask whether this role is fundamental, that is, a natural consequence of what positional analysis techniques are seeking to achieve, or simply incidental. By incidental we mean that the examples we deal with, both real and constructed, just happen (possible because they are numerous) to succumb successfully to techniques which employ centrality in their implementation. Returning directly to the Doreian technique, this question can be answered if we can find a graph in which everybody has the same centrality but there are clearly different positions within the network. In a simple cycle all points have the same centrality but they are all perfectly substitutable and consequently are playing the same role. It is not a simple task to find non-trivial graphs in which every individual has the same centrality. However, the graph in the next section is such a graph and we believe that it is a simple matter to identify two distinct positions for the vertices. The graph provides a tough test for any positional analysis technique and acts as a counter-example to a number of conjectures in the positional analysis literature.

2. The AVLF graph

In 1969 four soviet mathematicians found a counter-example to show that every distance-regular graph is not distance-transitive (for definitions see Biggs 1974). We shall call their example the AVLF graph, named after the authors Adelson-Velskii, Veisfeiler, Leman and Faradzev. In essence the AVLF graph consists of two distinct groups of nodes which share many combinatorial properties. Let Γ be the graph with the 26 vertices $\{a_0, a_1, \dots a_{12}, b_0, b_1, \dots b_{12}\}$ where

$$
\begin{array}{lr}
a_i \text{ and } a_j \text{ are adjacent iff } (i-j) \equiv 1, 3, 4, 9, 10, 12 & (\text{mod } 13) \\
b_i \text{ and } b_j \text{ are adjacent iff } (i-j) \equiv 2, 5, 6, 7, 8, 11 & (\text{mod } 13) \\
a_i \text{ and } b_j \text{ are adjacent iff } (i-j) \equiv 0, 1, 3, 9 & (\text{mod } 13)
\end{array}
$$

The adjacency matrix is given in Table 1.

Each vertex has degree 10 and the graph has diameter 2. Every vertex has the same centrality for all the standard centrality measures, i.e. betweenness, closeness, all degree-based measures, etc. In addition, the more revealing geodesic and dependency matrices on which these

Table 1

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	0	1	0	1	1	0	0	0	0	1	1	0	1	1	0	0	0	1	0	0	0	0	0	1	0	1
2	1	0	1	0	1	1	0	0	0	0	1	1	0	1	1	0	0	0	1	0	0	0	0	0	1	0
3	0	1	0	1	0	1	1	0	0	0	0	1	1	0	1	1	0	0	0	1	0	0	0	0	0	1
4	1	0	1	0	1	0	1	1	0	0	0	0	1	1	0	1	1	0	0	0	1	0	0	0	0	0
5	1	1	0	1	0	1	0	1	1	0	0	0	0	0	1	0	1	1	0	0	0	1	0	0	0	0
6	0	1	1	0	1	0	1	0	1	1	0	0	0	0	0	1	0	1	1	0	0	0	1	0	0	0
7	0	0	1	1	0	1	0	1	0	1	1	0	0	0	0	0	1	0	1	1	0	0	0	1	0	0
8	0	0	0	1	1	0	1	0	1	0	1	1	0	0	0	0	0	1	0	1	1	0	0	0	1	0
9	0	0	0	0	1	1	0	1	0	1	0	1	1	0	0	0	0	0	1	0	1	1	0	0	0	1
10	1	0	0	0	0	1	1	0	1	0	1	0	1	1	0	0	0	0	0	1	0	1	1	0	0	0
11	1	1	0	0	0	0	1	1	0	1	0	1	0	0	1	0	0	0	0	0	1	0	1	1	0	0
12	0	1	1	0	0	0	0	1	1	0	1	0	1	0	0	1	0	0	0	0	0	1	0	1	1	0
13	1	0	1	1	0	0	0	0	1	1	0	1	0	0	0	0	1	0	0	0	0	0	1	0	1	1
14	1	1	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	1	1	1	1	0	0	1	0
15	0	1	1	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	1	1	1	1	0	0	1
16	0	0	1	1	0	1	0	0	0	0	0	1	0	1	0	0	0	1	0	0	1	1	1	1	0	0
17	0	0	0	1	1	0	1	0	0	0	0	0	1	0	1	0	0	0	1	0	0	1	1	1	1	0
18	1	0	0	0	1	1	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0	0	1	1	1	1
19	0	1	0	0	0	1	1	0	1	0	0	0	0	1	0	0	1	0	0	0	1	0	0	1	1	1
20	0	0	1	0	0	0	1	1	0	1	0	0	0	1	1	0	0	1	0	0	0	1	0	0	1	1
21	0	0	0	1	0	0	0	1	1	0	1	0	0	1	1	1	0	0	1	0	0	0	1	0	0	1
22	0	0	0	0	1	0	0	0	1	1	0	1	0	1	1	1	1	0	0	1	0	0	0	1	0	0
23	0	0	0	0	0	1	0	0	0	1	1	0	1	0	1	1	1	1	0	0	1	0	0	0	1	0
24	1	0	0	0	0	0	1	0	0	0	1	1	0	0	0	1	1	1	1	0	0	1	0	0	0	1
25	0	1	0	0	0	0	0	1	0	0	0	1	1	1	0	0	1	1	1	1	0	0	1	0	0	0
26	1	0	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	1	1	1	1	0	0	1	0	0

measures are built also do not show any differences in pattern between any pair of vertices.

However, if we examine the first-order neighbourhoods of the a and b vertices, we find significant differences in structure. The first-order neighbourhoods (often called first-order stars) are simply the induced subgraphs formed by examining all vertices adjacent to a particular vertex. Figure 1 shows the neighbourhoods of "a" and "b" vertices with the central vertex removed. We note that the a's neighbourhood contains one clique of 3 vertices whereas b's neighbourhood contains 3 cliques of 3 vertices. It should be noted that the neighbourhoods have the same structure for any "a" or "b" vertex. It follows from this that the natural positional analysis partition of the vertices for the graph is therefore $\{a_0, a_1, \ldots, a_{12}\}$, $\{b_0, b_1, \ldots, b_{12}\}$.

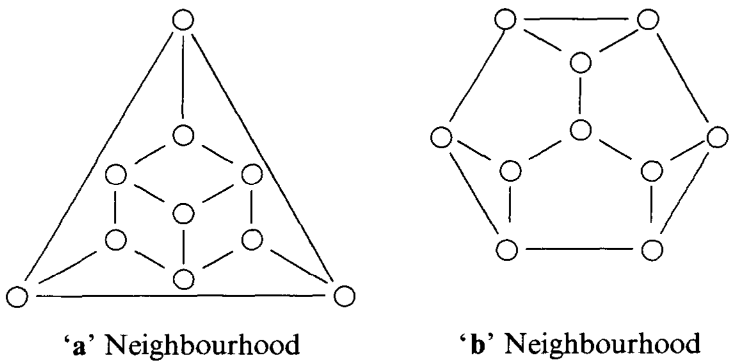

Fig. 1.

The AVLF graph was submitted to all the positional analysis routines contained in UCINET (i.e. CONCOR, Euclidean distance, regular equivalences, structural equivalences and semigroup analysis). None of these algorithms produce the natural partition (regular equivalence fails since the graph is symmetric). In addition the Doreian technique fails to partition the vertices (since all vertices have the same centrality) and we conclude that centrality is not sufficient to capture the role concept.

3. Three conjectures refuted

Borgatti (1988) conjectures that the Doreian technique is equivalent to finding the orbits of a graph. An automorphism of a graph $G(V, R)$ with vertex set V and edge set R is a permutation π of the vertices V which has the property that $(a, b) \in R$ if and only if $(\pi(a), \pi(b)) \in R$. The set of all automorphisms of G form a group under the operation of composition which is denoted by Aut(G). Two vertices $a, b \in V$ belong to the same orbit of G if and only if $\pi(a) = b$ for some $\pi \in$ Aut(G). The orbits of the AVLF graph are $\{a_0, a_1, \ldots, a_{12}\}$, $\{b_0, b_1, \ldots, b_{12}\}$. Since the Doreian technique fails to split the vertex set, the graph acts as a counter-example to this conjecture. It is also interesting to note that the algorithm proposed by Everett and Borgatti (1988) does find the orbits and hence is the only positional analysis technique which produces the natural partition of the graph.

Winship (1988) has recently published his often cited previously unpublished manuscript of 1974 "Thoughts about Roles and Relations". The paper contains a number of conjectures, two of which can be refuted by the AVLF graph. Let $G(V, R)$ be a graph with n vertices and adjacency matrix A. If $i, j \in V$ then the relational column rc(i, j) of vertex i with j is the countably infinite vector rc$(i, j) = (A_{ij}, A^2_{ij}, A^3_{ij}, \ldots)$.

The relational plane rp(i) of a vertex i is the infinite matrix rp$(i) = ($rc$(i, 1)$, rc$(i, 2), \ldots,$ rc$(i, n))$. Two vertices are congruent if their relational planes are equal to within a permutation of their columns. Winship's first conjecture is that two vertices belong to the same orbit if and only if they are congruent. (In his 1988 paper he adds a footnote to say that he believes the conjecture to be false, but does not provide a counter-example).

If two vertices belong to the same orbit they must have perfect substitutability and hence it follows that they are congruent; however, the converse is false. If we examine increasing powers of A it becomes apparent that for any power the entries consist of three different numbers. There is one value on the diagonal, a different value corresponding to adjacent vertices and a third value for non-adjacent vertices. It follows that each vertex has non-trivial relational columns (we ignore the column rc(i, i)), one corresponding to those that the vertex is adjacent to and those for which it is not adjacent. From this it follows that all the vertices are congruent and consequently, as anticipated by Winship, the conjecture is false.

In the same paper Winship gives the following definition. Two vertices a, b are automorphically equivalent with respect to c if and only if there exists an automorphism π such that $\pi(a) = b$ and $\pi(c) = c$. He conjectures that if rc$(i, j) =$ rc(i, k) then j and k are automorphically equivalent with respect to i. (Also in the later paper he states he believes the conjecture to be false). It is a simple matter to see that the conjecture is false if we take i to be an isolate and j and k any pair of vertices in separate orbits. In this case rc$(i, j) =$ rc$(i, k) = (0, 0, 0, \ldots)$ and we know that no automorphism exists which maps j to k. However, it may be that the conjecture assumed that the graphs were connected. We now give a counter-example to this conjecture.

Let H be the AVLF graph together with an additional vertex x connected to each vertex of the AVLF graph. Any automorphism of the AVLF graph will be an automorphism of H provided $\pi(x) = x$.

From the comments in the previous paragraph it follows that, for example, $\mathrm{rc}(x, a_0) = \mathrm{rc}(x, b_{12})$ but we know that a_0 and b_{12} are in different orbits and hence cannot be automorphically equivalent with respect to x.

4. Conclusion

The mathematical literature contains a wealth of examples which could be used by networkers to gain a better understanding of their techniques. Whilst these examples may never occur in real data they do provide idealised tests of formal definitions. It can be very instructive to search for or construct examples designed to defeat algorithms or definitions. These examples can clarify thinking and at the same time correct misconceptions. We would like to encourage researchers working on new concepts or algorithms to make a serious attempt to defeat their ideas, rather than show they work in simple well-structured data. The network community would benefit from this approach in the long run.

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