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Ecological and perfect colorings [1]

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A recent study (Hummon and Carley 1992) has indicated that one of the main research paths in the discipline of social networks is the study of roles and positions. Several key positional concepts have been elucidated, including regular colorings (White and Reitz 1983), automorphic colorings (Everett 1985), and structural colorings (Lorrain and White 1971). Regular colorings in particular have been seen as formalizations of the intuitive concept of social role. However, in this paper we suggest that regular colorings only incorporate one half of the intuitive role concept. The other half, which we formalize as *ecological* colorings, is equally important. We therefore introduce *perfect* colorings, which are both regular and ecological, and are consequently more complete models of the role concept. Perfect colorings have a number of interesting mathematical properties, which we describe briefly.

1. Role colorings

Everett and Borgatti (1991) have suggested that a natural way of describing positional concepts is in terms of graph colorings. Let $G(V,E)$ be a finite connected graph without multiple edges with vertex set V and edge set E. (All the results can easily be extended to disconnected graphs, digraphs and networks.) A *role coloring* C is an assignment of colors to vertices such that the pattern of ties among colors follows certain well-defined rules. We denote the color of a vertex v by $C(v)$. We refer to vertices colored the same as *role equivalent*. The *spectrum* $C(S)$ of a subset S of V is the set of distinct colors assigned to the elements in the S.

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[1] We are grateful to Phillip Bonacich and an anonymous reviewer for several outstanding criticisms that have greatly improved our paper. We note that the paper still contains a few points on which we disagree.

The *neighborhood* $N(v)$ of a vertex v is the set of vertices adjacent to v. Given a graph G with vertex set V and edge set E, an *automorphism* π of G is a 1–1 mapping of G onto itself such that $(a, b) \in E$ iff $(\pi(a), \pi(b)) \in E$. The set of all automorphisms of G form a group under the operation of composition which we shall denote by $\mathrm{Aut}(G)$. Two vertices a and b belong to the same orbit of a subgroup H of $\mathrm{Aut}(G)$ if there exists a $\pi \in H$ such that $\pi(a) = b$.

Definition 1. A coloring C is *structural* if $C(a) = C(b)$ iff $N(a) = N(b)$.

Structural colorings were proposed by Lorrain and White (1971) and Burt (1976) as models of jointly occupied social positions or statuses. The idea is that two vertices that are connected and not connected to exactly the same other vertices in the graph, are perfectly substitutable. Hence, any advantages or disadvantages attributable to being connected to exactly that combination of vertices should accrue equally to both vertices.

Definition 2. A coloring C is *automorphic* if $C(a) = C(b)$ iff a and b belong to the same orbit of a subgroup of $\mathrm{Aut}(G)$.

Automorphic colorings were proposed by Everett (1985) and Winship (1988). Automorphic colorings form classes of vertices which are structurally indistinguishable, but which are not necessarily connected to any of the same vertices.

Definition 3. A coloring C is *regular* if $C(a) = C(b)$ implies $C(N(a)) = C(N(b))$.

Regular colorings were proposed by White and Reitz (1983) as formalizations of the sociological notion of social role. They can be interpreted as requiring that regularly equivalent vertices (i.e. same colored) be connected to regularly equivalent vertices. Applied to social roles, the idea is that if two actors are both doctors, then they have the same kinds of relationships with their respective patients, nurses, suppliers, and so on.

Regular colorings form a lattice which includes both structural and automorphic colorings as members (Borgatti and Everett 1989).

2. Ecological colorings

In a recent paper on exchange networks, Borgatti and Everett (1992b) introduced the notion of *ecological* colorings, which they define as follows:

Definition 4. A coloring C is *ecological* if $C(N(a)) = C(N(b))$ implies $C(a) = C(b)$.

A coloring is ecological if a vertex's color is completely determined by the colors of its neighbors. Thus, the definition states that two vertices surrounded by the same collections of colors will themselves be colored the same. Examples of ecological colorings are given in Fig. 1 and Fig. 2. An example of a non-ecological coloring is given in Fig. 3.

The appellation 'ecological' is meant to suggest a coloring in which the environment of a vertex determines something about the vertex. We expect ecological colorings to prove useful in formalizing sociological theories in which the environment of an actor is thought to shape

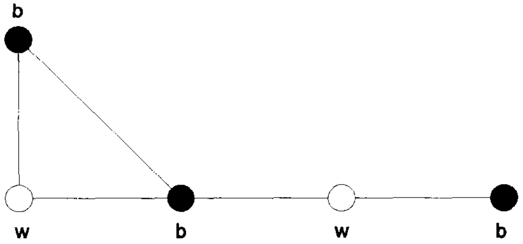

Fig. 1.

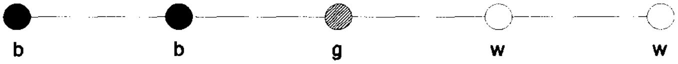

Fig. 2.

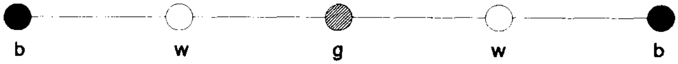

Fig. 3.

that actor. For example, in organizational theory, population ecologists (Hannan and Freeman 1977; Aldrich 1979) hypothesize that organizational forms are determined, through natural selection and adaptation, by their environments, which consist largely of other organizational forms. Similarly, in biology, it is commonplace to attribute features of a species' morphology to its relations (who eats whom, who shares resources with whom) with other species.

The ecological view is also prevalent in network theories of attitude formation (Erickson 1988) and diffusion of innovation (Burt 1987). In fact, it is fundamental to all views of contagion (Burt 1992) that are based on concepts of spatial autocorrelation (Cliff and Ord 1973). For example, Burt (1991) models an actor's attitude as a linear function of the average of the attitudes of those connected to the actor. Thus, two actors surrounded by the same combination of attitudes are predicted to have the same attitudes as each other.

One area in which ecological colorings have already proven useful is the analysis of power in experimental exchange networks. Borgatti and Everett (1992b) used ecological colorings to formalize the intuitive concept that a vertex's power is a function of the power of its neighbors. Defining a power coloring as an assignment of colors to vertices such that two vertices are assigned the same color if and only if they have the same power, they showed that all experimentally observed power colorings were ecological. That is, in all experiments, it turns out that if two vertices are surrounded by the same combination of powers, they are equally powerful. Borgatti and Everett also showed that the theory of Markovsky *et al.* (1988) always yields ecologically consistent predictions.

Ecological colorings of graphs are similar in spirit to eigenvectors. An eigenvector is a vector v that satisfies the following condition:

$$v_i = \lambda^{-1} \sum_j x_{ij} v_j$$

If X is a symmetric adjacency matrix, then v_i is the eigenvector score for vertex i. If the adjacency matrix contains only zeros and ones, the v_i is proportional to the sum of the scores of its neighbors. Hence, if two vertices have the same combinations of eigenvector scores in their neighborhoods, they will themselves have equal eigenvector scores. In this sense, ecological colorings may be thought of as simple precursors of something which might be called ‘qualitative eigenvectors’ [1].

In any graph, there may be a number of ecological colorings. These can be ordered by the refinement relation $\leqslant$. Any coloring induces a partition of the vertices into homogeneous classes with respect to color. Given two colorings C_1 and C_2 then $C_1 \leqslant C_2$ if every element of the induced partition of C_1 is a subset of an element of C_2. The following theorem demonstrates that the class of all ecological colorings of a graph has a similar structure to the class of regular colorings.

Theorem 5. The set of all ecological colorings of a graph, ordered by the refinement relation, form a lattice.

Proof. It is sufficient to prove the existence of arbitrary meets. The coloring in which every vertex is colored the same is obviously ecological and hence we need only consider the arbitrary meets of a non-empty set of ecological colorings. Let $C_1, \ldots, C_n$ be a set of ecological colorings for a graph G. Then let C be the coloring in which for any pair of vertices a, b we define C such that $C(a) = C(b)$ if and only if $C_i(a) = C_i(b)$ for $i = 1, \ldots, n$. It is well known that this construction gives the meet for any family of partitions: we need only show that it is ecological. Suppose that $C(N(a)) = C(N(b))$ so that for every $z \in N(a)$ there exists some $y \in N(b)$ such that $C(z) = C(y)$ and therefore $C_i(z) = C_i(y)$ for every i. Hence $C_i(N(a)) \subseteq C_i(N(b))$. Similarly, $C_i(N(b)) \subseteq C_i(N(a))$. It follows that

$$C_i(N(a)) = C_i(N(b)) \text{ for } i = 1, 2, \ldots, n$$

Since C_i is ecological for every i then

$$C_i(a) = C_i(b) \text{ for } i = 1, 2, \ldots, n$$

hence $C(a) = C(b)$ as required. □

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[1] As both Phillip Bonacich (personal communication) and an anonymous reviewer have cautioned us, the analogy cannot be pushed too far. An extension of the notion of ecological colorings might define the spectrum $C(S)$ of a set of vertices S as an array instead of a set (so that a vertex surrounded by two green vertices was different from one surrounded by three green vertices), the analogy would be closer. However, that extension is beyond the scope of this paper.

The minimal element of the ecological lattice is the *structural* coloring, in which vertices with identical neighborhoods are assigned the same color.

3. Ecological and regular colorings

A comparison of Definition 4 with Definition 3 reveals that ecological colorings are in a certain sense opposites of regular colorings. Whereas in a regular coloring the color of a vertex implies a certain combination of colors in its environment, in an ecological coloring it is the colors in the environment which determine the color of the vertex.

In some ways, the ecological coloring is easier to comprehend as a model of social relationships. We can easily imagine a process (e.g. attitude formation) by which the kinds of vertices in a particular vertex's neighborhood would tend to shape that vertex into this or that kind. Thus we can see the ecological coloring as the end state of an influence process. In contrast, a regular coloring is more difficult to associate with a social process. If we used the same kind of reasoning as with ecological colorings, we would posit that each vertex exerts an influence on its neighborhood which determines the distribution of colors therein. However, this is hardly plausible since each element in the neighborhood is also a neighbor of several other vertices as well, each exerting their own influence. Hence, a regular coloring requires positing a larger, network-wide force which simultaneously structures the patterning of all links. [2] Whereas an ecological coloring may be seen as emerging from the behavior of individual elements, regular colorings must come from the collectivity as a whole. In this sense, ecological colorings are more *local* or *micro* in spirit while regular colorings are more *global* or *macro*. [3]

Both notions fit our intuitive understanding of the social concept of role. A named social role such as 'doctor' or 'mother' carries with it a set of relationships with other roles that every actor playing the role is expected to have with individuals playing reciprocal roles. For exam-

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[2] For example, a regular coloring may be the result of cultural norms regarding rights of and obligations of individuals in certain positions vis-à-vis individuals in other positions.

[3] Of course, even ecological colorings are not as local as structural colorings. In a structural coloring, one can determine whether to color two vertices the same without simultaneously coloring all other vertices in the graph. This is not true of ecological or regular colorings.

ple, we are unpleasantly surprised when we find a doctor that does not heal patients or a mother that does not care for her children. This understanding is consistent with the notion of a regular coloring. At the same time, we expect that if an individual's relationships with others are identical to those expected for a given role, then they can be considered to be playing that role. For example, if a person takes on all the duties of a teacher with respect to a set of students, it would be hard to avoid recognizing that they are playing the role of teacher. This understanding is consistent with an ecological coloring.

4. Perfect colorings

While ecological and regular colorings may be opposites in a certain sense, many colorings are both ecological and regular. We call such colorings *perfect.* They may be defined as follows:

Definition 6. A coloring C is *perfect* if $C(a) = C(b)$ iff $C(N(a)) = C(N(b))$.

According to the definition, a coloring is perfect if it is both regular and ecological. That is, vertices with the same color environments are themselves the same color, and vertices of the same color have the same color environments. An example of a perfect coloring is given in Fig. 4. An example of a regular coloring that is not perfect was given in Fig. 3.

The color image graph $G'(C(V), E')$ of a colored graph $G(V, E)$ has the spectrum of V as its vertices; two vertices are adjacent in G' if there exists an edge between the colors in G. Figure 5 is the color image graph of the non-perfect coloring given in Fig. 3. Note that the

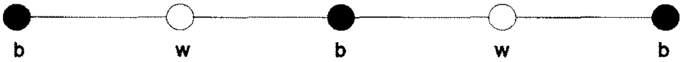

Fig. 4.

Fig. 5.

image graph contains structurally equivalent vertices. As we shall see, an important characteristic of a perfect coloring is that its image graph contains *no* structurally equivalent vertices. In fact, this is the basis of the alternative definition of perfect coloring given by the next theorem. *Note:* The proof of the theorem requires the result given by Borgatti (1989) that a coloring is regular if and only if for every vertex $a \in V$, $C(N(a)) = N(C(a))$, where $C(a)$ is a vertex in the colored image graph (see Everett and Borgatti 1993 for the proof).

Theorem 7. A regular coloring is perfect if and only if the color image graph contains no structurally equivalent vertices.

Proof. Suppose a graph G is perfectly colored and that in the image graph $C(a)$ and $C(b)$ are two distinct structurally equivalent vertices. By definition $N(C(a)) = N(C(b))$ but since the coloring is regular, by the result above, $C(N(a)) = C(N(b))$. But the coloring is also ecological so that $C(a) = C(b)$, contradicting the fact that $C(a)$ and $C(b)$ are distinct. Conversely, suppose that G is regularly colored and the color image graph does not contain structurally equivalent vertices. If $C(N(a)) = C(N(b))$ then $N(C(a)) = N(C(b))$, so that in the color image graph $C(a)$ and $C(b)$ are structurally equivalent. It follows that $C(a) = C(b)$, and the theorem is complete. □

Theorem 7 highlights the superiority of a perfect coloring over a regular coloring. Consider a 3-block regular blockmodel (Borgatti and Everett 1992a) of an adjacency matrix, as shown in Fig. 6. The blockmodel is not perfect. Note that, at the block level, the first and second blocks have identical patterns of connections with other blocks. If the pattern of relations among blocks defines the role that members of each block play (Boorman and White 1976), then we must say that

	1	2	3	4	5	6	7
1					1	1	1
2					1		
3						1	
4							1
5	1	1					
6	1		1				
7	1			1			

0	0	1
0	0	1
1	1	0

Fig. 6. Regular blockmodel.

members of both blocks are playing the same role. If this is the case, it seems arbitrary and hardly useful to separate these actors into separate blocks. What Theorem 7 says is that perfect blockmodels are regular blockmodels in which all blocks with identical relations to other blocks are merged together. This means that, in perfect blockmodels, the blocks are, in a certain sense, as large as possible, which is a very desirable quality.

A maximal coloring is the coarsest partition which is consistent with a particular set of definitional conditions. The maximal regular coloring is the maximum element of the lattice of regular colorings. This corresponds to the regular partition found by REGE (White 1984; Borgatti *et al.* 1992). The maximal structural coloring is the coloring in which any pair of structurally equivalent vertices are colored the same. The maximal automorphic coloring corresponds to the coloring of the orbits of the automorphism group of the graph. An application of Theorem 7 yields the following corollaries.

Corollary 8. The maximal structural coloring is perfect.

Corollary 9. The maximal regular coloring is perfect.

The proofs of these follow from results proved by Borgatti *et al.* (1989). They show that the color image graph of a maximal structural coloring does not contain structurally equivalent vertices, [4] from which Corollary 8 follows. In the same paper they prove that the composition of regular colorings is regular. From this it follows that the image graph of the maximal regular coloring cannot contain structurally equivalent vertices (since this would induce a coarser regular coloring of the original graph) and Corollary 9 is established. An extremely useful consequence of Corollary 9 is that existing algorithms for detecting the maximal regular coloring of a graph need not be modified to find that particular perfect coloring. [5]

Unlike the others, the maximal automorphic equivalence is not necessarily perfect as shown by the example in Fig. 3.

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[4] They use a slightly different definition of structural equivalence than is used here, but the result is still valid.

[5] However, Borgatti and Everett (1993) have recently shown that some algorithms which purport to find the maximal regular equivalence, such as the REGE algorithm (White 1984; Borgatti *et al.* 1992), do not always do so.

Theorem 7 also gives us a method by which we can construct the nearest coarser perfect coloring to any regular coloring. Suppose a graph G has been regular colored and the color image graph contains some structurally equivalent vertices. The maximal structural coloring of the color image graph will induce the nearest coarser perfect coloring on the original graph. This result follows directly from Theorem 7 and the composition theorem for regular colorings (Borgatti *et al.* 1989). This result can be seen by considering Figs. 3, 4 and 5. Figure 3 gives a regular coloring. Figure 5 gives its color image graph. As noted earlier, the coloring in Fig. 3 is not perfect since the vertices in Fig. 5 colored black and grey are structurally equivalent. If we now color these vertices with the same color, such as black, then we induce the perfect coloring (of the original graph) that is shown in Fig. 4.

Since the composition of regular colorings is regular it follows from Theorem 7 that the composition of perfect colorings is perfect. In fact, any regular coloring followed by a perfect coloring will be perfect, although a perfect coloring followed by a regular coloring need not be perfect. Ecological colorings do not compose in this way.

It should come as no surprise that the class of all perfect colorings for a graph also forms a lattice under the refinement ordering. This result is proved in our final theorem.

Theorem 10. The set of all perfect colorings of a graph, ordered by the refinement relation, forms a lattice.

Proof. We shall prove the existence of arbitrary meets. For any set of perfect colorings we first construct the meet of all the colorings, using the same meet operation as in the lattice of regular colorings. This construction yields a regular coloring which may or may not be perfect. If it is perfect then we take this as the meet of the perfect colorings. If it is not perfect we find the nearest coarser meet using the construction described above based upon Theorem 7. The result is obviously a perfect coloring which is the infimum with respect to the refinement ordering. □

While all perfect colorings are regular, the lattice of perfect colorings is not a sub-lattice of the regular lattice: they have different joins. Consider, for example, the regular and perfect lattices associated with the graph in Fig. 7. Both lattices contain the partitions {{1, 2}, {4, 5},

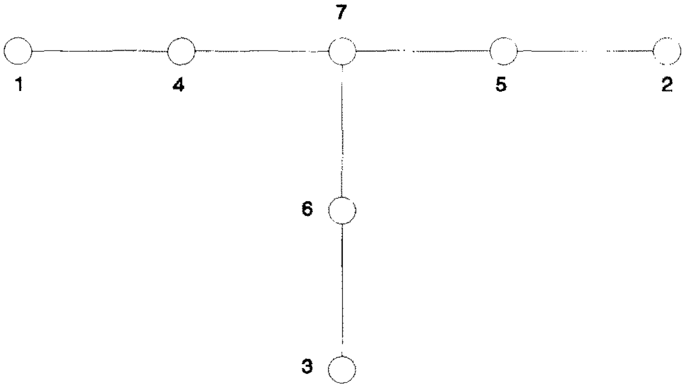

Fig. 7.

{3}, {6}, {7}} and {{2, 3}, {5, 6}, {1}, {4}, {7}}. In the regular lattice, the join of these two is {{1, 2, 3}, {4, 5, 6}, {7}}. But this partition is not perfect. In the perfect lattice, the join is {{1, 2, 3, 7}, {4, 5, 6}}. Hence, the perfect lattice is not a sub-lattice of the regular lattice.

5. Conclusion

White and Reitz (1983) have suggested that the intuitive notion of social role implies that if two actors are playing the same role, then they have the same social relations with the same players of the same roles. This is the property of regularity. However, there is more to the notion of social role than regularity. An equally important component of the role concept is the idea that if two actors have the same relations with the same roles, then they are themselves playing the same role. This is the property we have termed ecological. Thus, social roles are best formalized by perfect colorings, which possess both properties. In addition, the ecological property may be useful in other contexts as well, as in population ecology approaches to organizational structure, and power in exchange networks.

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