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NOTE

ROLE COLOURING A GRAPH

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The role colouring of a graph is an assignment of colours to the vertices which obeys the rule that two vertices are coloured the same only if their neighbourhoods have the same colour set. We investigate the set of role colourings for a graph proving that it forms a lattice. We also show that this lattice can be trivial and this can only occur if the graph has a trivial automorphism group.

Key words: Graph; lattice; automorphism group.

1. Introduction

Graph theory has been used as a model in the social sciences for some time; unfortunately this use has often been descriptive and has therefore not provoked interesting mathematical questions. The model is simple; the vertices of a graph represent individuals and the edges represent relationships between individuals. However, this model does become mathematically interesting when social scientists ask questions about structure - the most natural question relates to the definition of social role. Individuals play the same role if they relate in the same way to other individuals playing counterpart roles. White and Reitz (1983) formalise this concept in terms of graph homomorphisms and vertex partitions; they call their formalisation *regular equivalence*. In this paper we present an alternative definition of the same concept using ideas of vertex colouring. Since the word regular is over-used in mathematics (we shall encounter both regular graphs and regular permutation groups) we propose to use the term *role*. Let $G(V,E)$ be a finite connected graph without self loops or mulitple edges with vertex set V and edge set E. (All the results in this paper can easily be extended to disconnected graphs, multigraphs and digraphs.) The *neighbourhood* $N(v)$ of a vertex v is the set of all vertices adjacent to v. Suppose $G(V,E)$ is a graph such that each vertex $v \in V$ is assigned a colour $C(v)$ (note there is no rule as to how these colours are assigned), if $S \subset V$, then the *colour set* of S, $CS(S)$, is defined by

$$CS(S) = \{C(v) : v \in S\}.$$

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A *role colouring* of a graph $G(V,E)$ is an assignment of colours to the vertices with the property that for all $v_i, v_j \in V$

$$C(v_i) = C(v_j) \;\Rightarrow\; CS(N(v_i)) = CS(N(v_j)).$$

Every graph with more than one vertex has two *trivial role colourings*, namely when every vertex is a different colour and when every vertex is the same colour, any other role colouring is called a *non-trivial role colouring*. If $G(V,E)$ is bipartite with bipartition $\{V_1, V_2\}$, then colouring V_1 one colour and V_2 another is a role colouring.

The role colouring in Fig. 1 shows that non-bipartite graphs can be role coloured using just two colours. In Section 4 we prove that colouring each orbit a different colour produces a role colouring.

In any role colouring the set of vertices of a particular colour is called the *role-colour class*; the set of all role-colour classes is called the *role-colour partition*. Individuals in the same role-colour class are playing the same social role. In the graph in Fig. 1, individuals in role-colour class 1 only have relationships with 2's; the 2's however are connected to both 2's and 1's.

2. Simple results

Lemma 1. *Let $G(V,E)$ be a graph. Then in any non-trivial role colouring the role-colour set of the neighbourhood of a vertex cannot equal the role colour of the vertex, i.e. for all $v \in V$, $C(v) \neq CS(N(v))$.*

Proof. If for some vertex v, $C(v) = CS(N(v))$, then the same would need to be true for each vertex in $N(v)$. Hence, each vertex in successive neighbourhoods would be role coloured the same and since the graph is connected it follows that $CS(V) = C(v)$, contradicting the fact that the colouring is non-trivial. $\square$

Note that this result means that vertices adjacent to pendants must be coloured a different role colour than the pendant vertex. Hence, in any role colouring using two colours of a non-bipartite graph all pendants must belong to the same role-colour class.

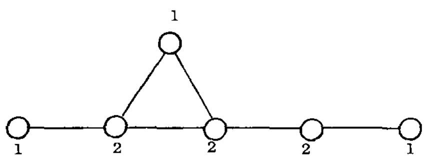

Fig. 1.

A graph in which any colouring is a role colouring is called *arbitrarily role-colourable.*

Lemma 2. *A graph is arbitrarily role-colourable if and only if it is K_n.*

Proof. It is obvious that K_n is arbitrarily role-colourable. Conversely, let $G(V,E)$ be a graph with two non-adjacent vertices, v_i and v_j. We colour $V-v_j$ with one colour and v_j with a different colour. By definition this is a non-trivial role colouring with v_i and $N(v_i)$ the same colour, contradicting the result of Lemma 1. □

3. The class of all role colouring

Let $G(V,E)$ be a graph. Then we denote the set of all role-colour partitions of $G(V,E)$ by $\mathbb{R}(G)$. We can order the elements of $\mathbb{R}(G)$ by the refinement relation $\leq$.

Theorem 3. *If $G(V,E)$ is a graph, then the set $\mathbb{R}(G)$ partially ordered by $\leq$ forms a complete lattice.*

Proof. We shall prove the existence of arbitrary joins; we note the trivial role-colour partitions provide us with a zero (and a unit) so we need only consider non-empty subsets of $\mathbb{R}(G)$. Let $R \in \mathbb{R}(G)$, then the role-colour partition induces an equivalence relation $\equiv_R$ on V. If I is a non-empty subset of $\mathbb{R}(G)$, then we can identify a family of induced equivalence relations $\equiv_i$, for each $i \in I$. Define a new relation $\equiv$ on V by $v_k \equiv v_j$ iff there exists a sequence $z_0, z_1, \dots, z_n$ with $v_k = z_0$, $v_j = z_n$, such that for all j in the range $1 \leq j \leq n, \exists i_j \in I$ such that $z_{j-1} \equiv_{i_j} z_j$. In other words, in our new colouring, two vertices, v_k and v_j, are in the same colour class if we can find a sequence of vertices beginning with v_k and ending with v_j such that every successive pair in the sequence is in the same colour class for some role colouring. We shall show that $\equiv$ induces a role colouring on G which is equal to $\vee I$. The construction of $\equiv$ is the same as that used in the construction of the join for the lattice of equivalence relations. Consequently, it is well known that $\equiv$ is an equivalence relation and a supremum with respect to the refinement ordering. We need only show that it induces a role colouring.

Suppose $v_k \equiv v_j$, with corresponding sequence $z_0, z_1, \dots, z_n$, and further suppose that $x \in N(v_k)$, hence $C(x) \in CS(N(v_k))$. Now $z_0 \equiv_{i_1} z_1$ and since $z_0 = v_k$, then $x \in N(z_0)$. It follows that $C_{i_1}(x) \in CS_{i_1}(N(z_1))$ and therefore $\exists d_1 \in N(z_1)$ with $C_{i_1}(x) = C_{i_1}(d_i)$ and hence $x \equiv_{i_1} d_1$. Similarly, since $z_1 \equiv_{i_2} z_2$ and $d_1 \in N(z_1)$ we can repeat the above argument and find a d_2 such that $d_2 \equiv_{i_2} d_1$. Continuing inductively we construct a sequence $x, d_1, d_2, \dots, d_n$, where $d_n \in N(z_n) = N(v_j)$ with each pair of the sequence in the same colour class for some member of I. It therefore follows

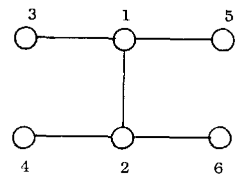

Fig. 2.

that $x \equiv d_n$ so that $C(x) \in CS(N(v_j))$ and hence $CS(N(v_k)) \subset CS(N(v_j))$. Similarly, $CS(N(v_j)) \subset CS(N(V_k))$ and the result follows. $\square$

It is not the case that $\mathbb{R}(G)$ is simply a sublattice of $\mathscr{E}(V)$, the lattice of all equivalence relations on V. Whilst the joins are constructed the same way, the meets are not. Consider the graph in Fig. 2. The vertex partitions $\{\{1,2\},\{3,4\},\{5,6\}\}$ and $\{\{1,2\},\{3,6\},\{4,5\}\}$ are both role colourings. The equivalence relation meet of these partitions would be $\{\{1,2\},\{3\},\{4\},\{5\},\{6\}\}$, which is not a role colouring.

4. Role primitive graphs

A graph with three or more vertices that only has trivial role colourings is said to be *role-primitive.* We first prove an existence theorem. Let H be the graph shown in Fig. 3.

Theorem 4. *The graph H is role-primitive.*

Proof. The proof is by contradiction. We first show that the vertices labelled 1 and 10 cannot be role coloured the same. Suppose $C(1) = C(10) = C_1$ (say). Since 1 and 10 are pendants, then by Lemma 1, 2 and 9 cannot be coloured by C_1. Suppose 2 and 9 are both coloured with C_2; since the graph is not bipartite, then 3 can only be coloured using C_2 or a new colour. If we colour 3 using C_2, then since we have coloured all the neighbours of a C_1 and a C_2 vertex, then we cannot introduce any new colours and we are forced to colour 4 with C_1, hence 5 and 6 must both be coloured with C_2, 7 with C_1 and 8 with C_2, which does not yield a role colouring. We therefore conclude that 3 must be a new colour, C_3. This means that 4 is forced to be C_3 and we are again in a position in which we are unable to introduce any new colours. Vertex 5 can now only be coloured using C_2 or C_3; suppose we use C_2. It follows that 6 must be coloured with C_1 forcing 7 to be coloured with C_2 and 8 with C_3; but now 8 must be adjacent to another vertex coloured C_3, which is impossible; hence, 5 must be coloured with C_3. We now have that 6 must be coloured with C_2, 7 with C_1 and 8 with C_2. Any vertex coloured C_2 must be adjacent to a C_1 and a C_3, hence 8 cannot be coloured with C_2. We are therefore forced to conclude

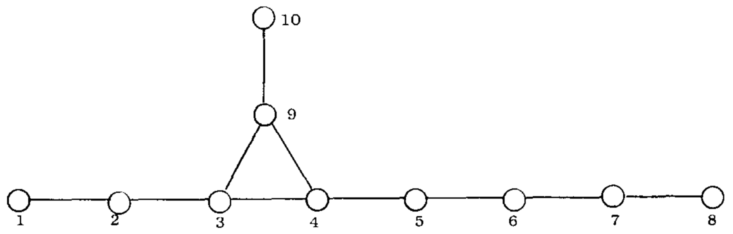

Fig. 3.

that 2 and 9 are coloured differently; but this is impossible if 1 and 10 have the same colour, hence 1 and 10 are coloured differently.

We complete the proof by showing that 1 and 10 cannot be role coloured differently. Suppose $C(1)=C_1$ and $C(10)=C_2$; again 2 and 9 must be differently coloured. Suppose 2 and 9 are both coloured with C_3, then we must have that $C(3)=C_2$ and $C(4)=C_1$. This is a contradiction since any vertex coloured C_1 can only be adjacent to vertices coloured C_3. It follows that 2 and 9 are coloured differently by, say, C_3 and C_4, respectively. Now 3 cannot be coloured by C_1 or C_2; suppose $C(3)=C_3$, then 4 must be coloured by C_1, a contradiction; if $C(3)=C_4$, then 4 would need to be coloured by C_2 and C_3 simultaneously. Hence 3 is coloured by a new colour, C_5, which implies that 4 can only be coloured by C_4, C_5 or a new colour. If $C(4)=C_4$, then $C(5)=C_2$ and hence $C(6)=C_4$, but any C_4 must be adjacent to a C_2, C_5 and another C_4, hence 6 cannot be coloured by C_4. Alternatively, if $C(4)=C_5$, then $C(5)=C_3$ so that $C(6)=C_1$ and hence $C(7)=C_3$ forcing 8 to be coloured with C_5, which is impossible.

We conclude that 4 must also be coloured by a new colour. It is now easy to see that all the remaining vertices must be coloured by a new colour and the theorem is proved. $\square$

The example above is the smallest role-primitive graph known to the authors. We note that it is an identity graph; our final theorem demonstrates that all role-primitive graphs are identity graphs. Note that the graph in Fig. 1 shows that the converse of this result is false.

Lemma 5. *Let $G(V,E)$ be a graph with automorphism group* $\mathrm{Aut}(G)$. *The orbits of any subgroup H of* $\mathrm{Aut}(G)$ *form a role-colour partition of V.*

Proof. If $C(v_i)=C(v_j)$ then there exists $\pi\in H$ st $\pi(v_i)=v_j$. If $x\in N(v_i)$, then $\pi(x)\in N(\pi(v_i))$ so that $\pi(x)\in N(v_j)$ but $C(x)=C(\pi(x))$, by definition. Hence $CS(N(v_i))\subset CS(N(v_j))$. The proof that $CS(N(v_j))\subset CS(N(v_i))$ is similar. $\square$

Theorem 6. *If $G(V,E)$ is role-primitive, then G is an identity graph.*

Proof. If G is role-primitive, then by the lemma either Aut(G) is the identity or Aut(G) acts transitively. If Aut(G) acts transitively, then again by the lemma the stabilizers must be trivial so that Aut(G) acts regularly. Since no subgroup of a regular group can be transitive, Aut(G) cannot contain subgroups. It therefore follows that Aut(G) is of prime order and so is Abelian. But the only Abelian automorphism groups which can act regularly on the vertices of a graph are the elementary Abelian 2-groups. Hence $\mathrm{Aut}(G) = Z_2$, contradicting the fact that G must have 3 or more vertices. □

The nature of this proof may lead the reader to suspect that the complement of a role-primitive graph is role-primitive, or at least contains few role colourings. This is not the case, the complement of the graph H has over 1000 different role colourings.

Reference

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