Fundamental concepts and operations

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GRAPH PROPERTIES

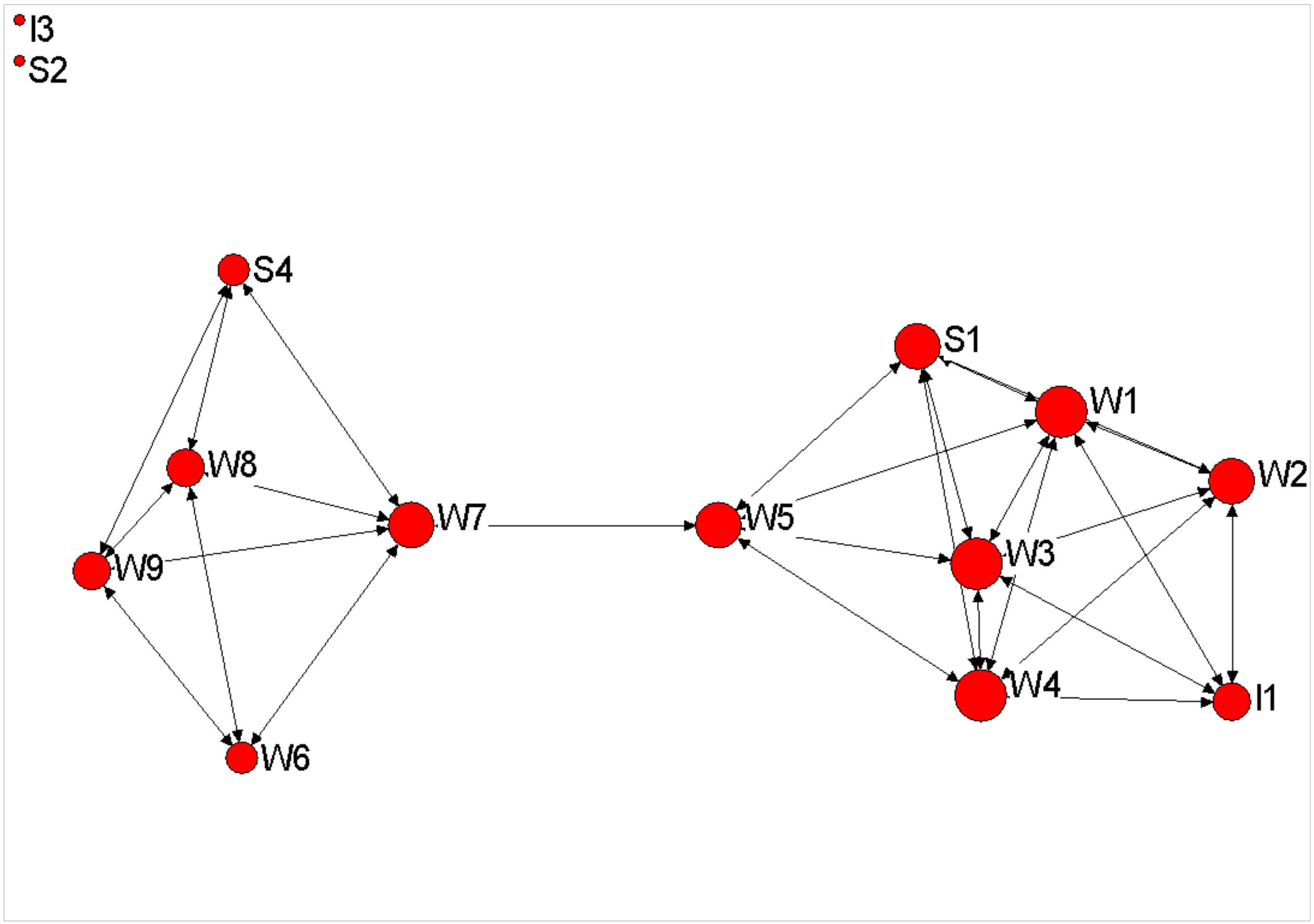
Degree

- Number of ties that involve a given node
  - Marginals (row/col sums) of symmetric adjacency matrix

Bank Wiring room data

	I1	I3	W1	W2	W3	W4	W5	W6	W7	W8	W9	S1	S2	S4	Deg
I1	0	0	1	1	1	1	0	0	0	0	0	0	0	0	4
I3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
W1	1	0	0	1	1	1	1	0	0	0	0	1	0	0	6
W2	1	0	1	0	1	1	0	0	0	0	0	1	0	0	5
W3	1	0	1	1	0	1	1	0	0	0	0	1	0	0	6
W4	1	0	1	1	1	0	1	0	0	0	0	1	0	0	6
W5	0	0	1	0	1	1	0	0	1	0	0	1	0	0	5
W6	0	0	0	0	0	0	0	0	1	1	1	0	0	0	3
W7	0	0	0	0	0	0	1	1	0	1	1	0	0	1	5
W8	0	0	0	0	0	0	0	1	1	0	1	0	0	1	4
W9	0	0	0	0	0	0	0	1	1	1	0	0	0	1	4
S1	0	0	1	1	1	1	1	0	0	0	0	0	0	0	5
S2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S4	0	0	0	0	0	0	0	0	1	1	1	0	0	0	3

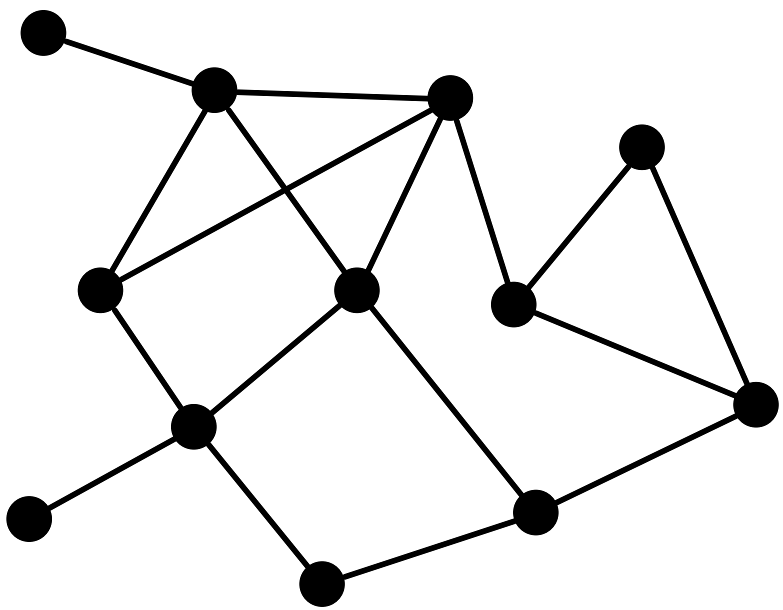
Wiring/Games Degree



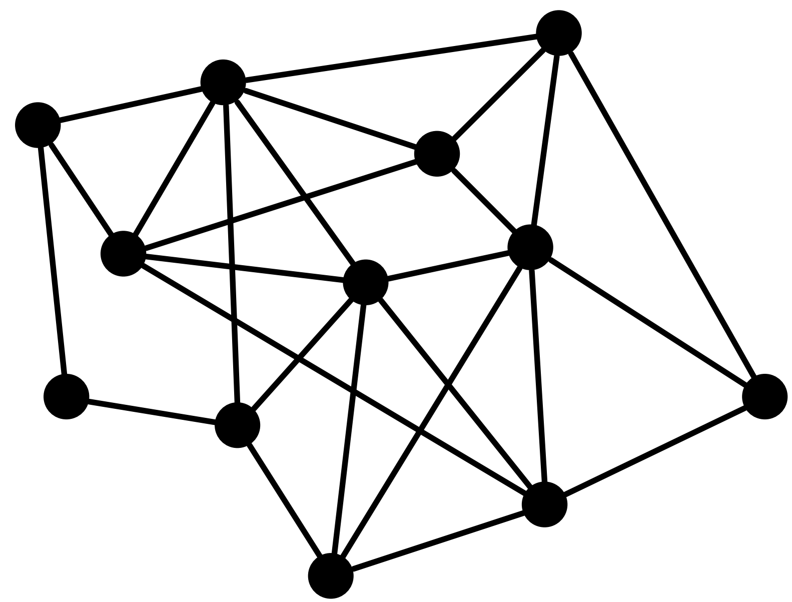
	Deg
I1	4
I3	0
W1	6
W2	5
W3	6
W4	6
W5	5
W6	3
W7	5
W8	4
W9	4
S1	5
S2	0
S4	3

Density

- Number of ties, expressed as percentage of the number of ordered/unordered pairs



Low Density (25%)
Avg. Dist. = 2.27



High Density (39%)
Avg. Dist. = 1.76

Density

Number of ties divided by number possible

	Reflexive	Non-Reflexive
Undirected	$= \frac{T}{n^2/2}$	$= \frac{T}{n(n-1)/2}$
Directed	$= \frac{T}{n^2}$	$= \frac{T}{n(n-1)}$

T = number of ties in network
n = number of nodes

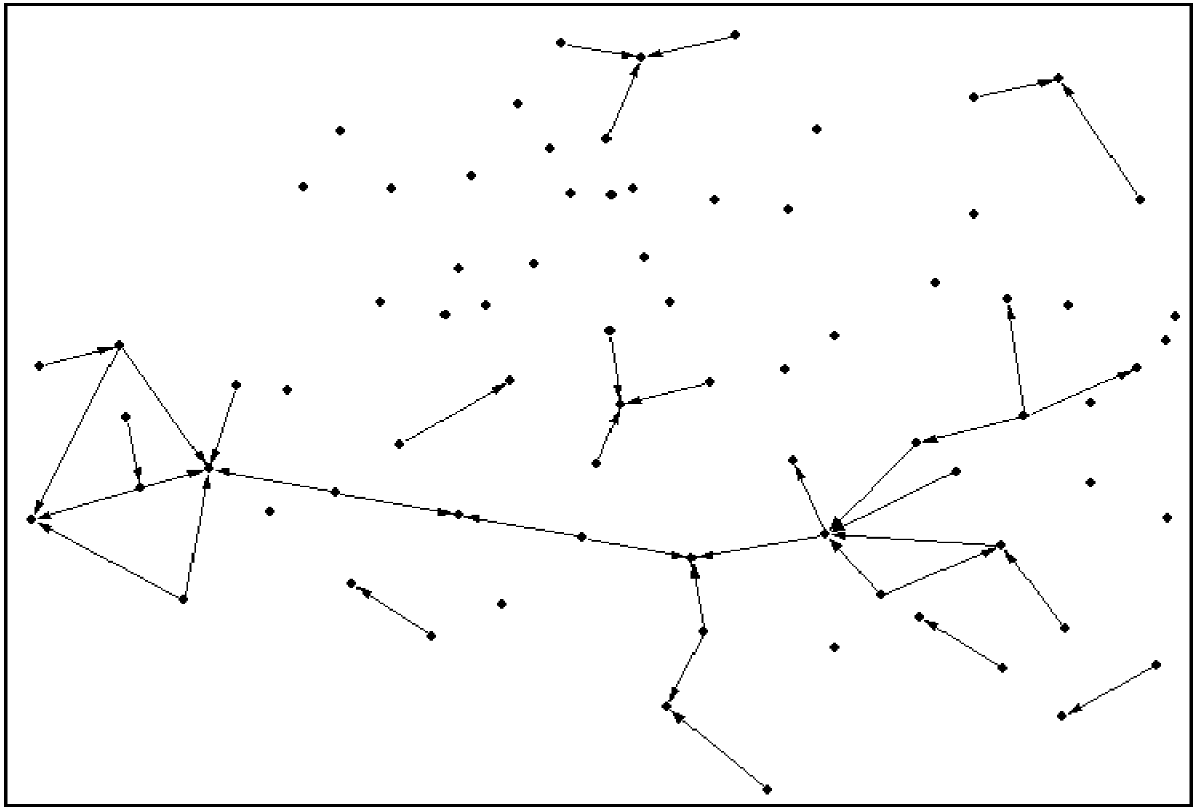
Density as aggregated dyadic cohesion

	HOLLY	BILL	DON	HARRY	MICHAEL	PAM	JENNIE	ANN	PAULINE	PAT	CAROL	LEE	JOHN	BRAZEY	GERY	STEVE	BERT	RUSS	Avg
HOLLY		0	1	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0.294
BILL	0		1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0.176
DON	1	1		1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0.235
HARRY	1	1	1		1	0	0	0	0	0	0	0	0	0	0	0	0	0	0.235
MICHAEL	1	1	1	1		0	0	0	0	0	0	0	0	0	1	0	0	0	0.294
PAM	1	0	0	0	0		1	1	1	0	1	0	0	0	0	0	0	0	0.294
JENNIE	0	0	0	0	0	1		1	0	1	0	0	0	0	0	0	0	0	0.176
ANN	0	0	0	0	0	1	1		1	0	0	0	0	0	0	0	0	0	0.176
PAULINE	0	0	0	0	0	1	0	1		1	1	0	1	0	0	0	0	0	0.294
PAT	1	0	0	0	0	0	1	0	1		1	0	0	0	0	0	0	0	0.235
CAROL	0	0	0	0	0	1	0	0	1	1		0	0	0	0	0	0	0	0.176
LEE	0	0	0	0	0	0	0	0	0	0	0		0	1	0	1	1	0	0.176
JOHN	0	0	0	0	0	0	0	0	1	0	0	0		0	1	0	0	1	0.176
BRAZEY	0	0	0	0	0	0	0	0	0	0	0	1	0		0	1	1	0	0.176
GERY	0	0	0	0	1	0	0	0	0	0	0	0	1	0		1	0	1	0.235
STEVE	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1		1	1	0.294
BERT	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1		1	0.235
RUSS	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1		0.235
Avg	0.29	0.18	0.24	0.24	0.29	0.29	0.18	0.18	0.29	0.24	0.18	0.18	0.18	0.18	0.24	0.29	0.24	0.24	0.229

Four Views

- Number of ones divided by number of valid cells
- Sum of all values divided by number of valid cells
- Avg of all values
- Avg of the row averages

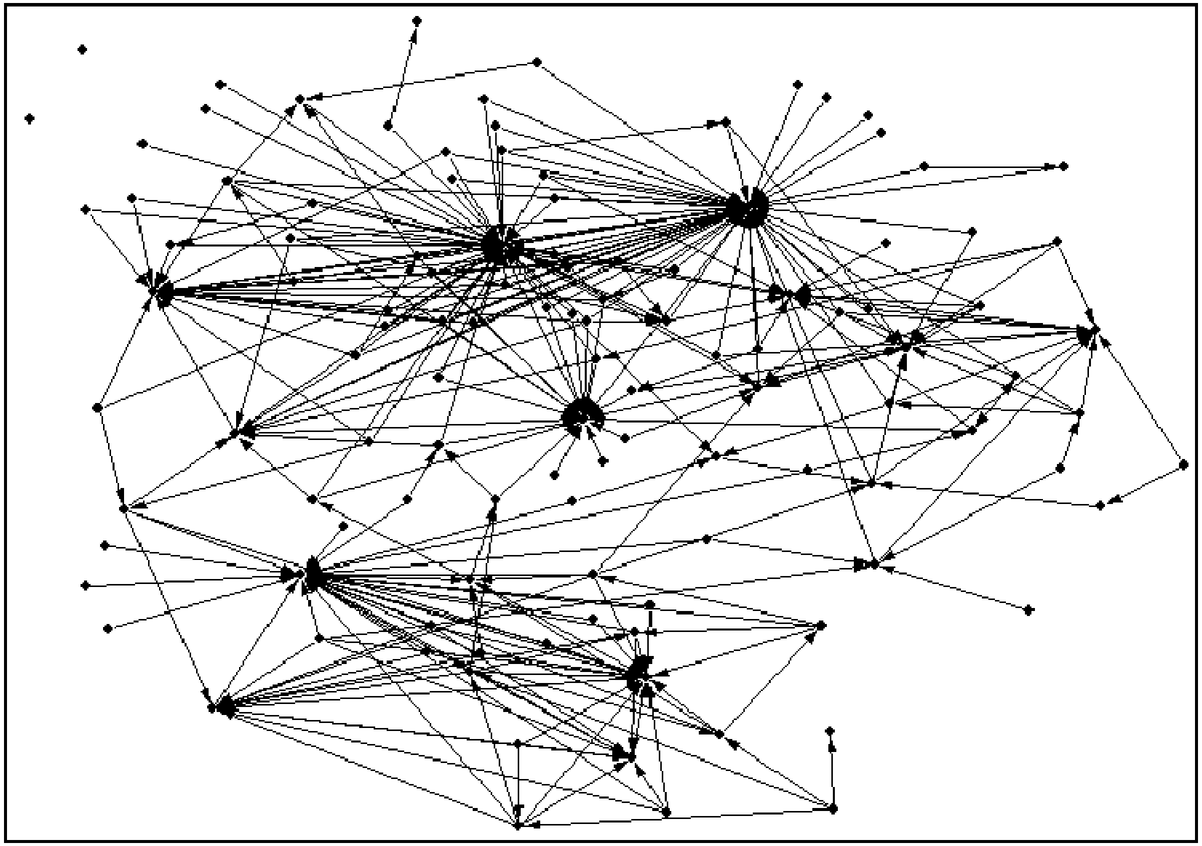
Help With the Rice Harvest



Village 1

Data from Entwistle et al

Help With the Rice Harvest



Village 2

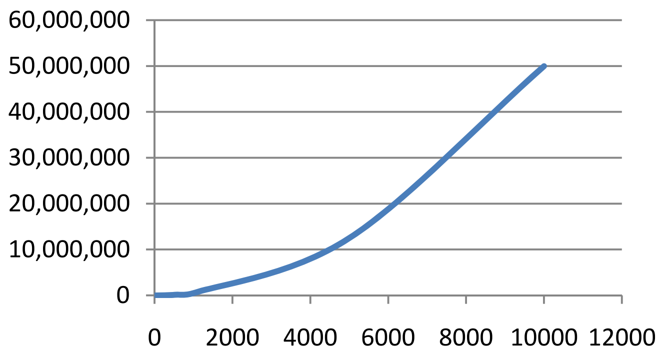
Which village is more likely to survive?

Data from Entwistle et al

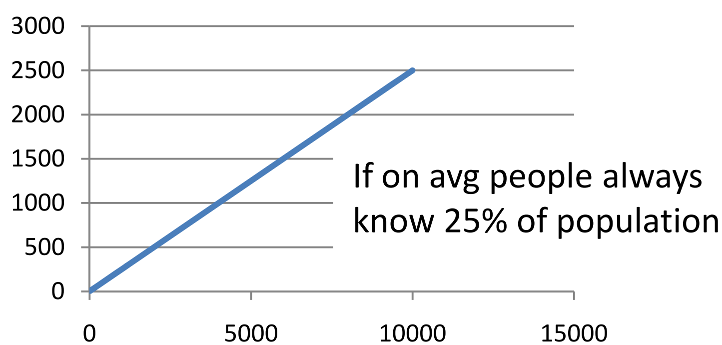
Density in large networks

$$\frac{\#\, ties}{\#\ possible} \approx \frac{\#\, ties}{\#\, node\ pairs}$$

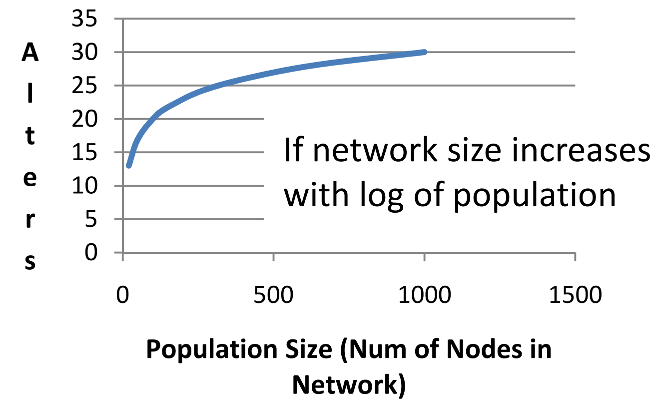
Pairs



Avg Number of Alters



Number of Alters

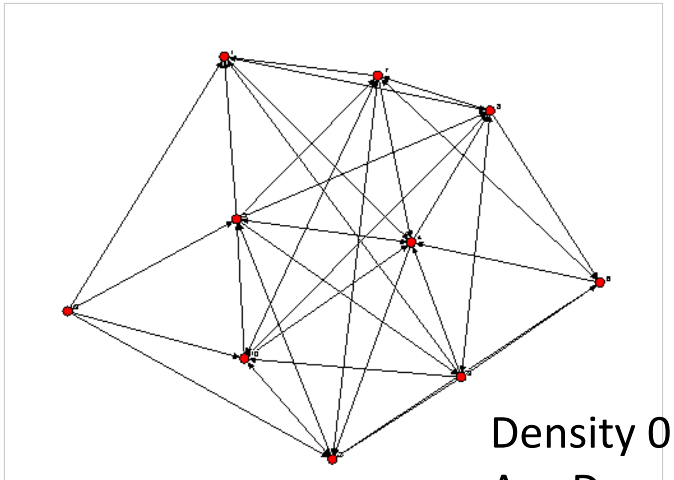


Network Density as a function of Number of Alters

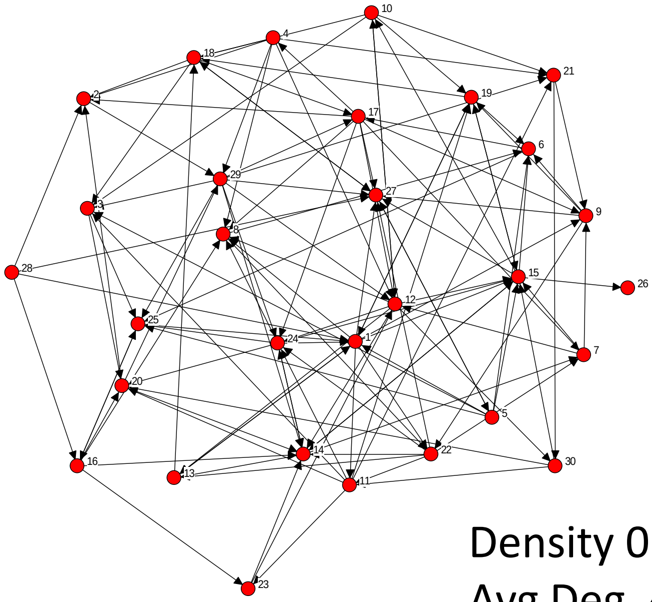
	Typical # of Alters		
Population	.25P	.25p+10	log(p)
20	0.263	0.789	0.685
50	0.255	0.459	0.347
100	0.253	0.354	0.202
150	0.252	0.319	0.146
300	0.251	0.284	0.083
600	0.250	0.267	0.046
1000	0.250	0.260	0.030
5000	0.250	0.252	0.007
10000	0.250	0.251	0.004

Average Degree

- Average number of links per person
- Is same as density*(n-1), where n is size of network
  - Density is just normalized avg degree – divide by max possible
- Often more intuitive than density



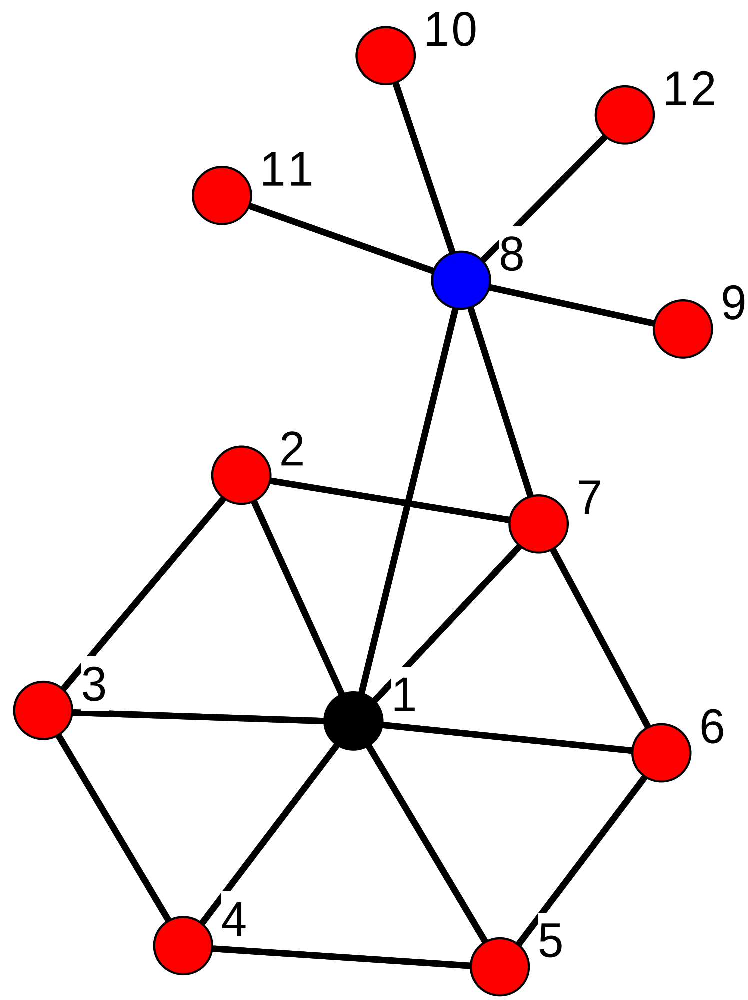
Density 0.47
Avg Deg 4



Density 0.14
Avg Deg 4

Walks, Trails, Paths

- Path: can't repeat node
  - 1-2-3-4-5-6-7-8
  - Not 7-1-2-3-7-4
- Trail: can't repeat line
  - 1-2-3-1-7-8
  - Not 7-1-2-7-1-4
- Walk: unrestricted
  - 1-2-3-1-2-7-1-7-1



Modeling flow processes

- Paths
  - Virus. Hosts become immune or die, so can never return to previous node
  - Con man working a group
- Trails
  - Gossip
  - Used paperback passed along – may be unknowingly be given to someone who already had it
- Walks
  - Dollar bill moving through economy

Components

- Maximal sets of nodes in which every node can reach every other by some path (no matter how long)
- A connected graph has just one component

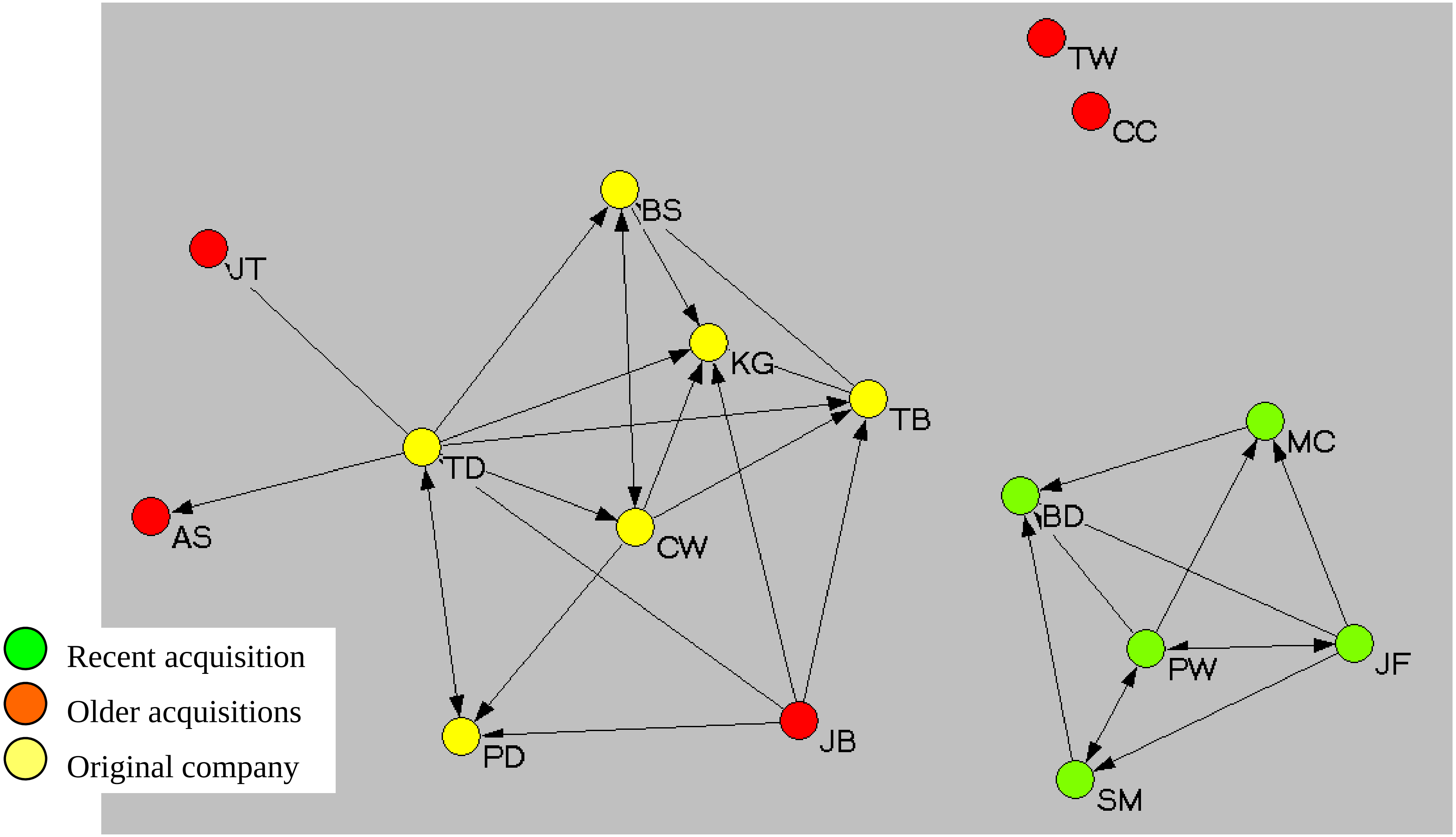
It is relations (types of tie) that define different networks, not components. A network that has two components remains one (disconnected) network.

Components in Directed Graphs

- Strong component
  - There is a directed path from each member of the component to every other
- Weak component
  - There is an undirected path (a weak path) from every member of the component to every other
  - Is like ignoring the direction of ties – driving the wrong way if you have to

A network with 4 components

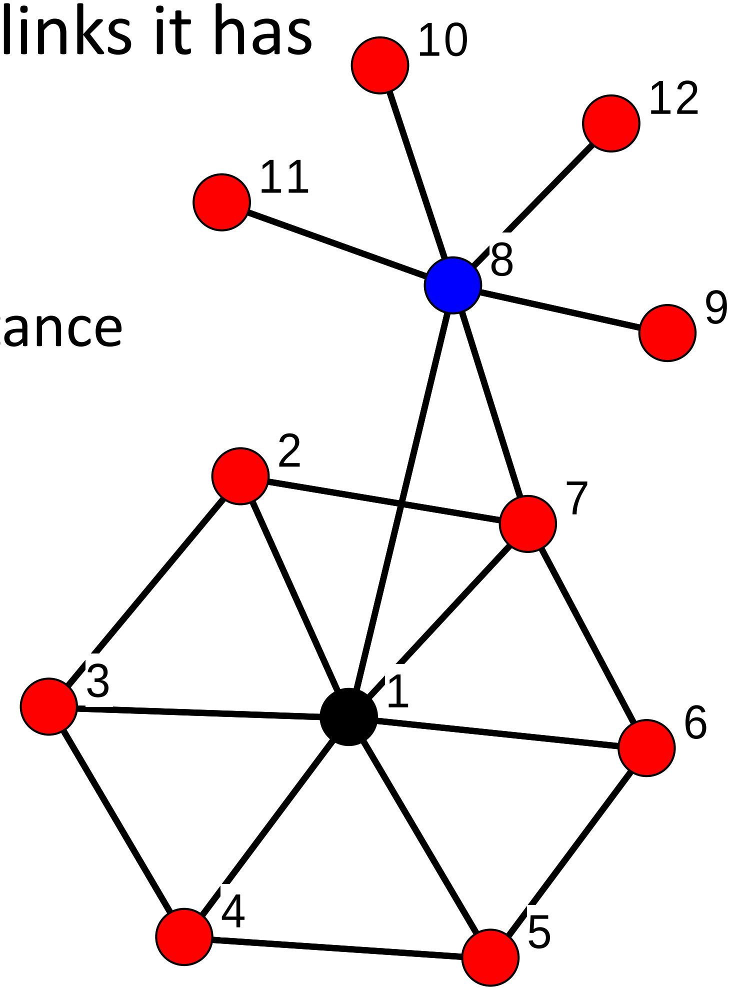
Who you go to so that you can say ‘I ran it by ____, and she says ...’



Data drawn from Cross, Borgatti & Parker 2001.

Length & Distance

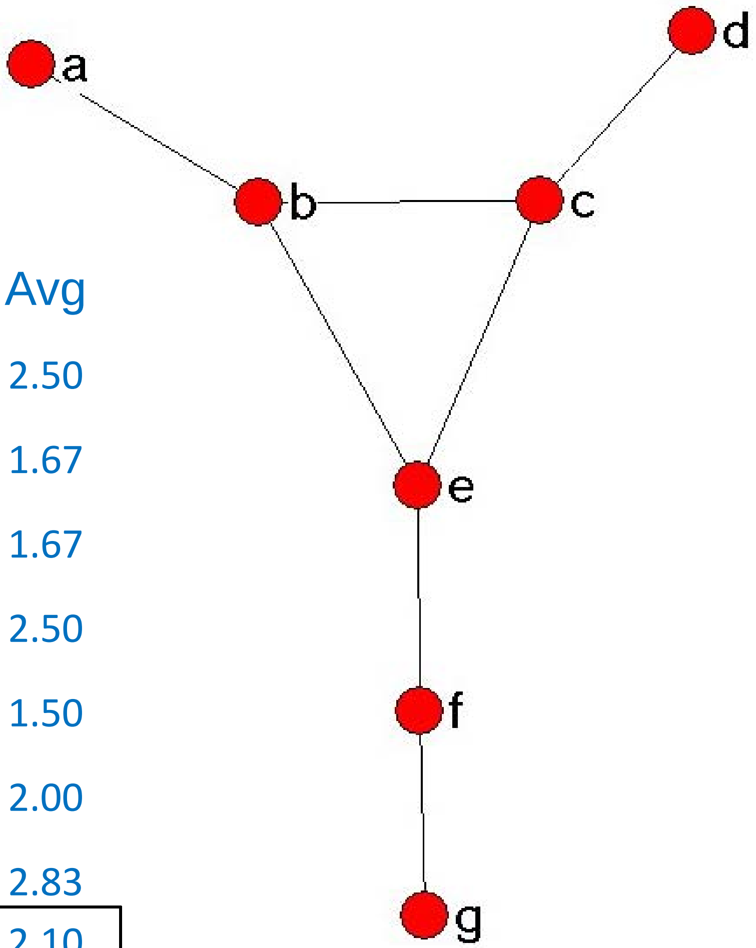
- Length of a path is number of links it has
- Distance between two nodes is length of shortest path
  - aka geodesic path, geodesic distance



http://oracleofbacon.org/

Geodesic Distance Matrix

	a	b	c	d	e	f	g	Avg
a		1	2	3	2	3	4	2.50
b	1		1	2	1	2	3	1.67
c	2	1		1	1	2	3	1.67
d	3	2	1		2	3	4	2.50
e	2	1	1	2		1	2	1.50
f	3	2	2	3	1		1	2.00
g	4	3	3	4	2	1		2.83
								2.10



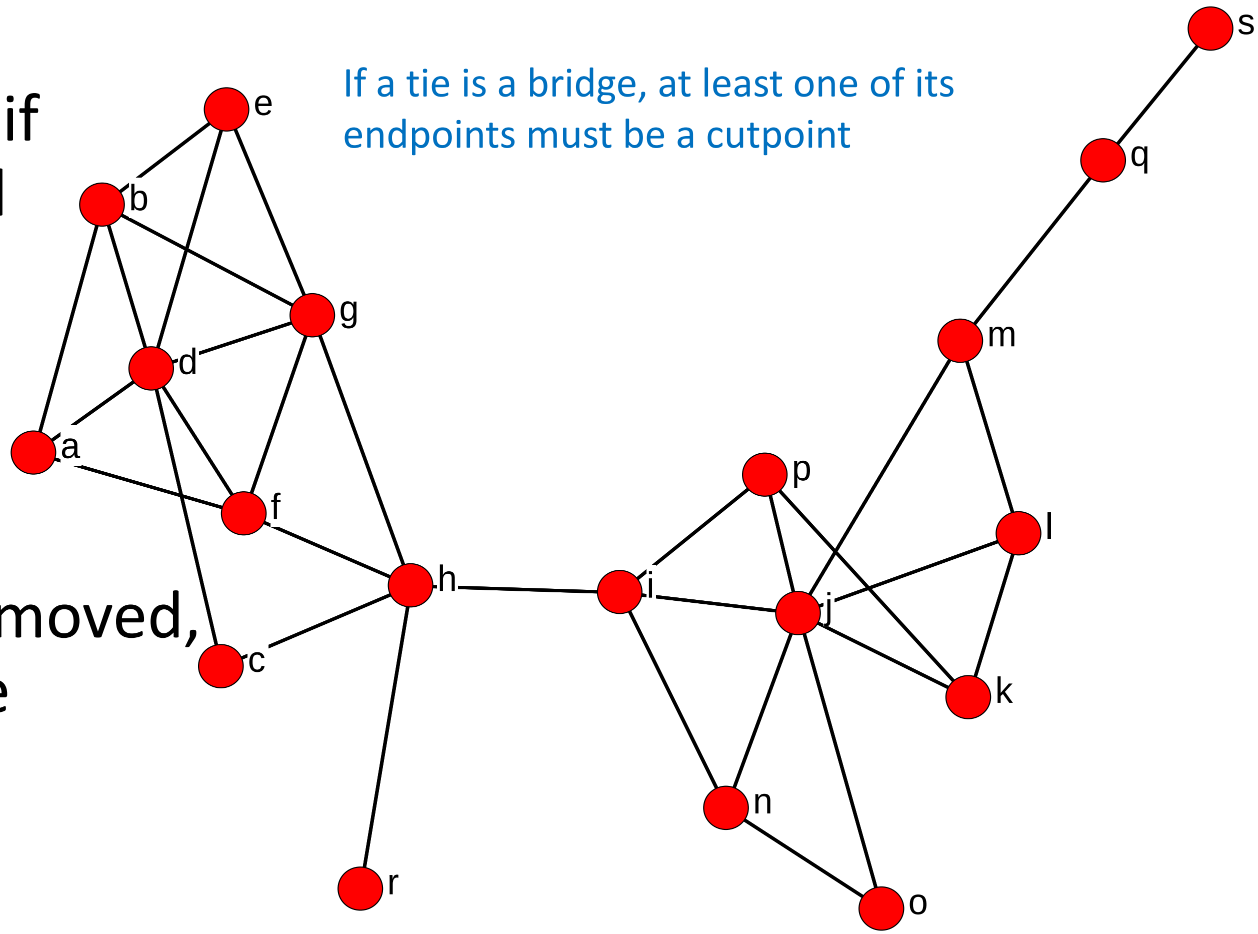
Implications of Distance

- For something flowing across links, expect distance to correlate …
  – Inversely with probability of arrival
  – Inversely with time until arrival
  – Positively with amount of distortion or change
- Averaging to node level we have
  – Typical distance to other nodes in network
- Averaging to network level we have
  – A type of cohesion

Cutpoints and Bridges

- Cutpoint
  - A node which, if deleted, would increase the number of components
- Bridge
  - A tie that, if removed, would increase the number of components

If a tie is a bridge, at least one of its endpoints must be a cutpoint



MATRIX OPERATIONS

Dichotomizing

- X is a valued matrix, say 1 to 5 rating of strength of tie
- Construct a matrix Y of ones and zeros so that
  – Yij = 1 if xij > 3, and Yij = 0 otherwise

	EVE	LAU	THE	BRE	CHA
EVELYN	8	6	7	6	3
LAURA	6	7	6	6	3
THERESA	7	6	8	6	4
BRENDA	6	6	6	7	4
CHARLOTTE	3	3	4	4	4

X

	EVE	LAU	THE	BRE	CHA
EVELYN	1	1	1	1	0
LAURA	1	1	1	1	0
THERESA	1	1	1	1	0
BRENDA	1	1	1	1	0
CHARLOTTE	0	0	0	0	0

Xij > 5

Symmetrizing

- When matrix is not symmetric, i.e., $X_{ij} \neq X_{ji}$
- Symmetrize various ways. Set Y_{ij} and Y_{ji} to:
  - Maximum(X_{ij}, X_{ji}) {union rule}
  - Minimum (X_{ij}, X_{ji}) {intersection rule}
  - Average: $(X_{ij} + X_{ji})/2$
  - Lowerhalf: choose X_{ij} when $i > j$ and X_{ji} otherwise
  - etc

Symmetrizing Example

- X is non-symmetric (and happens to be valued)
- Construct matrix Y such that Yij (and Yji) = maximum of Xij and Xji

	ROM	BON	AMB	BER	PET	LOU
ROMUL_10	0	1	1	0	3	0
BONAVEN_5	0	0	1	0	3	2
AMBROSE_9	0	1	0	0	0	0
BERTH_6	0	1	2	0	3	0
PETER_4	0	3	0	1	0	2
LOUIS_11	0	2	0	0	0	0

X

	ROM	BON	AMB	BER	PET	LOU
ROMUL_10	0	1	1	0	3	0
BONAVEN_5	1	0	1	1	3	2
AMBROSE_9	1	1	0	2	0	0
BERTH_6	0	1	2	0	3	0
PETER_4	3	3	0	3	0	2
LOUIS_11	0	2	0	0	2	0

Symmetrized by Maximum

Transpose

- Interchange rows and columns

	Age	Gender	Income
Mary	32	1	90,000
Bill	50	2	45,000
John	12	2	0
Larry	20	2	8,000

	Mary	Bill	John	Larry
Age	32	50	12	20
Gender	1	2	2	2
Income	90,000	45,000	0	8,000

Transpose Adjacency matrix

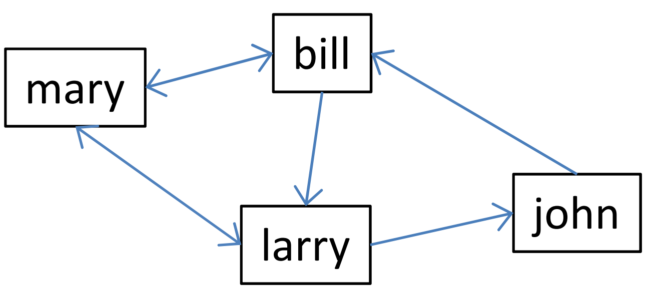
- Interchanging rows/columns of adjacency matrix effectively reverses the direction of ties

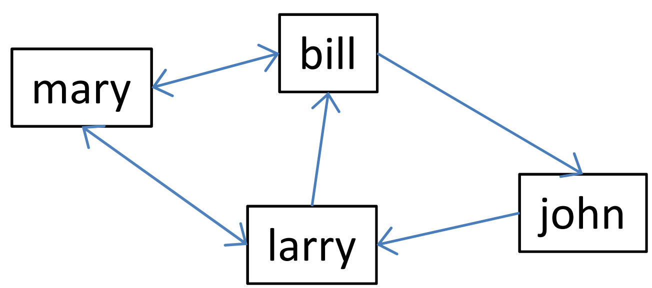
	Mary	Bill	John	Larry
Mary	0	1	0	1
Bill	1	0	0	1
John	0	1	0	0
Larry	1	0	1	0

Gives money to

	Mary	Bill	John	Larry
Mary	0	1	0	1
Bill	1	0	1	0
John	0	0	0	1
Larry	1	1	0	0

Gets money from





Marginals

	Mary	Bill	John	Larry	Row Marginals
Mary	0	1	1	1	3
Bill	1	0	1	0	2
John	0	0	0	1	1
Larry	0	0	0	0	0
Column Marginals	1	1	2	2	6

Matrix marginal

Matrix Product

- Notation: C = AB

- Definition: $$c_{ij} = \sum_k a_{ik} b_{jk}$$

- Example:

	Mary	Bill	John	Larry
Mary	0	1	1	1
Bill	1	0	1	0
John	0	0	0	1
Larry	0	0	0	0

A

	Mary	Bill	John	Larry
Mary	0	0	1	1
Bill	1	0	1	0
John	0	0	0	1
Larry	0	1	0	0

B

	Mary	Bill	John	Larry
Mary	1	1	1	1
Bill	0	0	1	2
John	0	1	0	0
Larry	0	0	0	0

C=AB

Multiplying a matrix by itself

- A^2=AA
- Cell a_{ij}^2 gives the number of walks of length *2* from *i* to *j*
- In general a_{ij}^k gives the number of walks of length *k* from *i* to *j*

A

	Mary	Bill	John	Larry
Mary	0	0	1	1
Bill	1	0	1	0
John	0	0	0	1
Larry	0	1	0	0

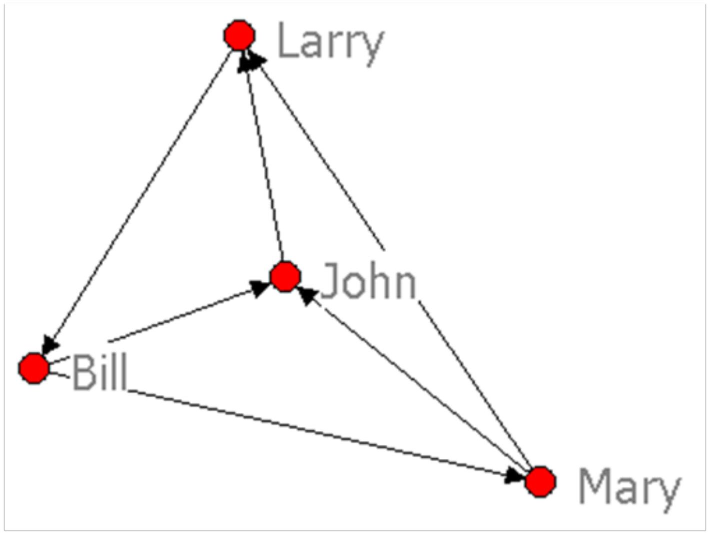
$\times$ A

	Mary	Bill	John	Larry
Mary	0	0	1	1
Bill	1	0	1	0
John	0	0	0	1
Larry	0	1	0	0

= A^2

	Mary	Bill	John	Larry
Mary	0	1	0	1
Bill	0	0	1	2
John	0	1	0	0
Larry	1	0	1	0

Powers of a Matrix



	Mary	Bill	John	Larry
Mary	0	0	1	1
Bill	1	0	1	0
John	0	0	0	1
Larry	0	1	0	0

A

	Mary	Bill	John	Larry
Mary	0	1	0	1
Bill	0	0	1	2
John	0	1	0	0
Larry	1	0	1	0

A^2

	Mary	Bill	John	Larry
Mary	1	1	1	0
Bill	0	2	0	1
John	1	0	1	0
Larry	0	0	1	2

A^3

	Mary	Bill	John	Larry
Mary	1	0	2	2
Bill	2	1	2	0
John	0	0	1	2
Larry	0	2	0	1

A^4

	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10	E11	E12	E13	E14
EVELYN	1	1	1	1	1	1	0	1	1	0	0	0	0	0
LAURA	1	1	1	0	1	1	1	1	0	0	0	0	0	0
THERESA	0	1	1	1	1	1	1	1	1	0	0	0	0	0
BRENDA	1	0	1	1	1	1	1	1	0	0	0	0	0	0
CHARLOTTE	0	0	1	1	1	0	1	0	0	0	0	0	0	0
FRANCES	0	0	1	0	1	1	0	1	0	0	0	0	0	0
ELEANOR	0	0	0	0	1	1	1	1	0	0	0	0	0	0
PEARL	0	0	0	0	0	1	0	1	1	0	0	0	0	0
RUTH	0	0	0	0	1	0	1	1	1	0	0	0	0	0
VERNE	0	0	0	0	0	0	1	1	1	0	0	1	0	0
MYRNA	0	0	0	0	0	0	0	1	1	1	0	1	0	0
KATHERINE	0	0	0	0	0	0	0	1	1	1	0	1	1	1
SYLVIA	0	0	0	0	0	0	1	1	1	1	0	1	1	1
NORA	0	0	0	0	0	1	1	0	1	1	1	1	1	1
HELEN	0	0	0	0	0	0	1	1	0	1	1	1	0	0
DOROTHY	0	0	0	0	0	0	0	1	1	0	0	0	0	0
OLIVIA	0	0	0	0	0	0	0	0	1	0	1	0	0	0
FLORA	0	0	0	0	0	0	0	0	1	0	1	0	0	0

Multiplying a matrix by its transpose

	EVELYN	LAURA	THERESA	BRENDA	CHARLOTTE	FRANCES	ELEANOR	PEARL	RUTH	VERNE	MYRNA	KATHERINE	SYLVIA	NORA	HELEN	DOROTHY	OLIVIA	FLORA
E1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
E2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
E3	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
E4	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
E5	1	1	1	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0
E6	1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0
E7	0	1	1	1	1	0	1	0	1	1	0	0	1	1	1	0	0	0
E8	1	1	1	1	0	1	1	1	1	1	1	1	1	0	1	1	0	0
E9	1	0	1	0	0	0	0	1	1	1	1	1	1	1	0	1	1	1
E10	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0
E11	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1
E12	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0
E13	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
E14	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0

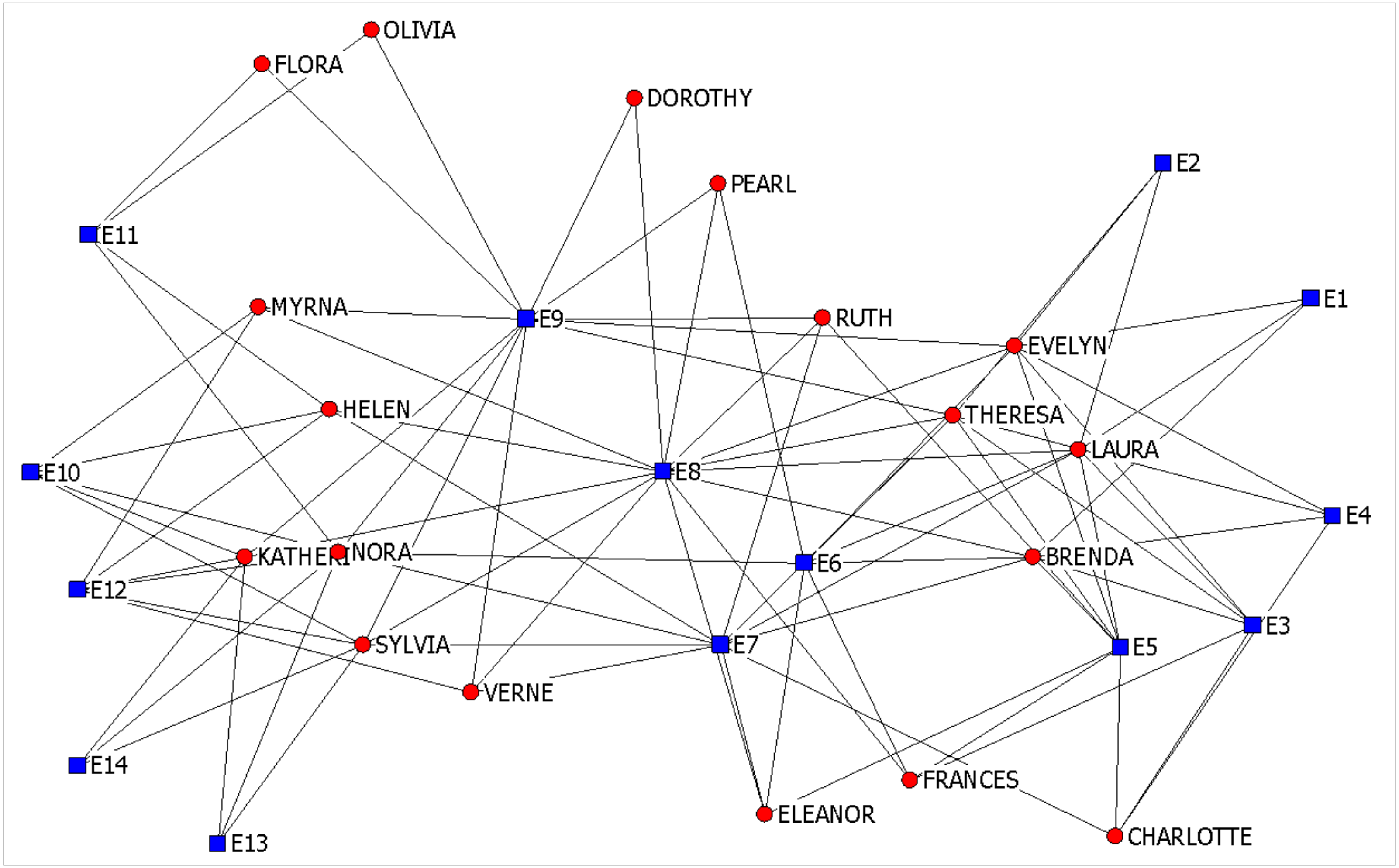
Woman by Woman Matrix

	EVE	LAU	THE	BRE	CHA	FRA	ELE	PEA	RUT	VER	MYR	KAT	SYL	NOR	HEL	DOR	OLI	FLO
EVELYN	8	6	7	6	3	4	3	3	3	2	2	2	2	2	1	2	1	1
LAURA	6	7	6	6	3	4	4	2	3	2	1	1	2	2	2	1	0	0
THERESA	7	6	8	6	4	4	4	3	4	3	2	2	3	3	2	2	1	1
BRENDA	6	6	6	7	4	4	4	2	3	2	1	1	2	2	2	1	0	0
CHARLOTTE	3	3	4	4	4	2	2	0	2	1	0	0	1	1	1	0	0	0
FRANCES	4	4	4	4	2	4	3	2	2	1	1	1	1	1	1	1	0	0
ELEANOR	3	4	4	4	2	3	4	2	3	2	1	1	2	2	2	1	0	0
PEARL	3	2	3	2	0	2	2	3	2	2	2	2	2	2	1	2	1	1
RUTH	3	3	4	3	2	2	3	2	4	3	2	2	3	2	2	2	1	1
VERNE	2	2	3	2	1	1	2	2	3	4	3	3	4	3	3	2	1	1
MYRNA	2	1	2	1	0	1	1	2	2	3	4	4	4	3	3	2	1	1
KATHERINE	2	1	2	1	0	1	1	2	2	3	4	6	6	5	3	2	1	1
SYLVIA	2	2	3	2	1	1	2	2	3	4	4	6	7	6	4	2	1	1
NORA	2	2	3	2	1	1	2	2	2	3	3	5	6	8	4	1	2	2
HELEN	1	2	2	2	1	1	2	1	2	3	3	3	4	4	5	1	1	1
DOROTHY	2	1	2	1	0	1	1	2	2	2	2	2	2	1	1	2	1	1
OLIVIA	1	0	1	0	0	0	0	1	1	1	1	1	1	2	1	1	2	2
FLORA	1	0	1	0	0	0	0	1	1	1	1	1	1	2	1	1	2	2

Bipartite Graph representation

Bipartite Graph representation

Bipartite Graph representation

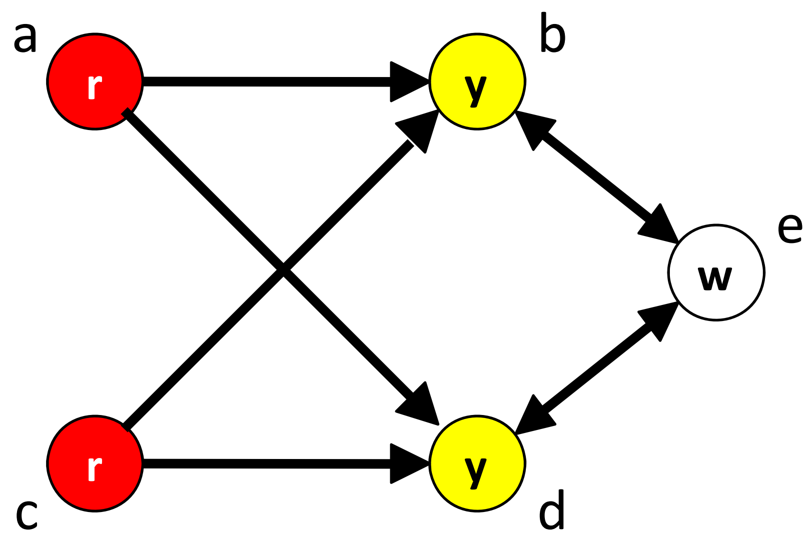


Adjacency matrix for bipartite graph

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0
1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	0	1	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	0	1	1	1	1	1	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	1	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

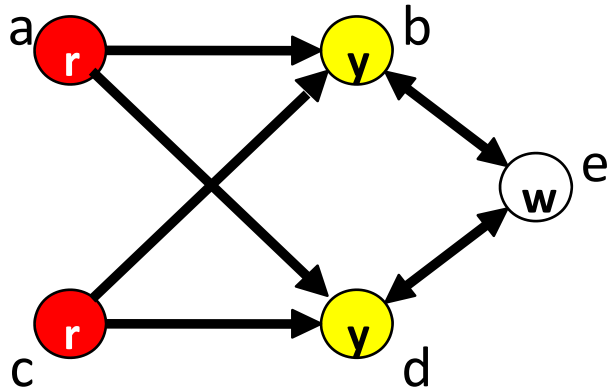
Structural Equivalence

- Structurally equivalent nodes have the same types of ties with the same third parties



Note: Equivalent nodes have been colored the same.

Structural Equivalence



	a	b	c	d	e
a	-	1	0	1	0
b	0	-	0	0	1
c	0	1	-	1	0
d	0	0	0	-	1
e	0	1	0	1	-

Adjacency matrix

	a	b	c	d	e
a	1	0.333	1	0.333	0.667
b	0.333	1	0.333	1	0.333
c	1	0.333	1	0.333	0.667
d	0.333	1	0.333	1	0.333
e	0.667	0.333	0.667	0.333	1

Structural Equivalence