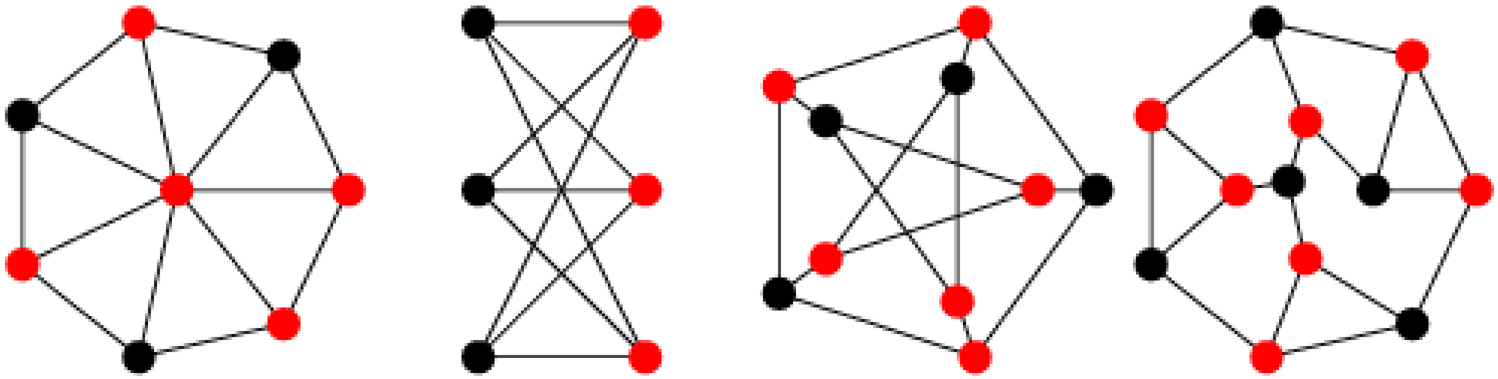


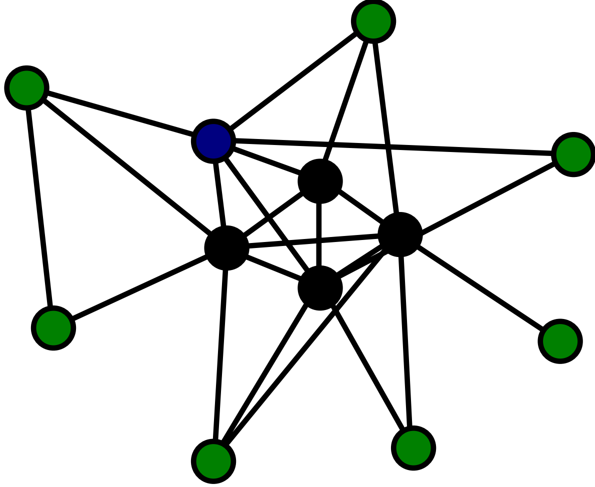
Mathematical Foundations

Steve Borgatti

Binary Relations

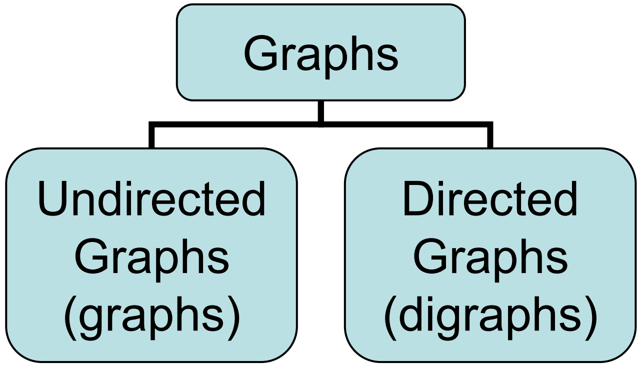
- A relation R is a collection of ordered pairs (u,v)
- If $(u,v) \in R$ this means u has relation R with v
  - If R is "gives advice to", then $(u,v) \in R$ means u gives advice to v
- If R is symmetric then $(u,v) \in R$ implies $(v,u) \in R$ for all u,v
- If R is reflexive then $(u,u) \in R$ for all u

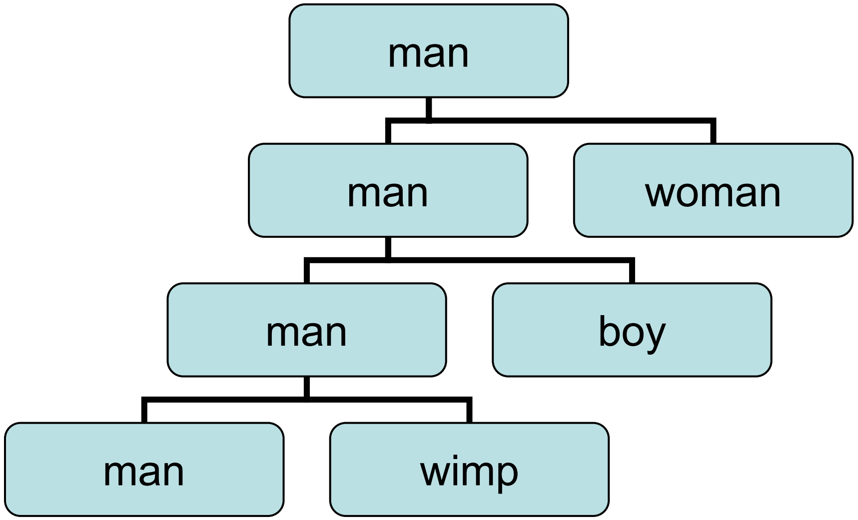
Graphs



- A graph G(V,E) consists of …
  - Set V of nodes|vertices|points representing actors
  - Set E of lines|edges representing ties
    - An edge is an unordered pair of nodes (u,v)
    - Nodes u and v adjacent if $(u,v) \in E$
    - So E is subset of set of all pairs of nodes
- The set E of unordered pairs is similar to a symmetric relation
  - But E has only half as many “data points” to represent same information

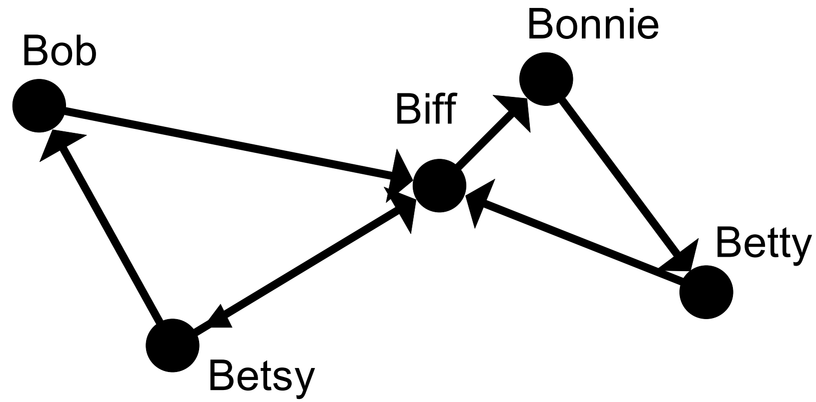
Directed & Undirected





Digraphs

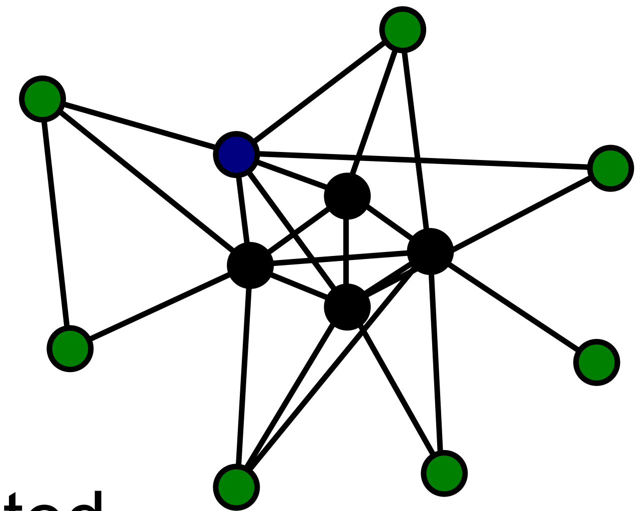
- Digraph D(V,E) consists of …
  - Set of nodes V
  - Set of directed arcs E
    - An arc is an ordered pair of nodes (u,v)
    - (u,v) ∈ E indicates u sends arc to v
    - (u,v) ∈ E does not imply that (v,u) ∈ E

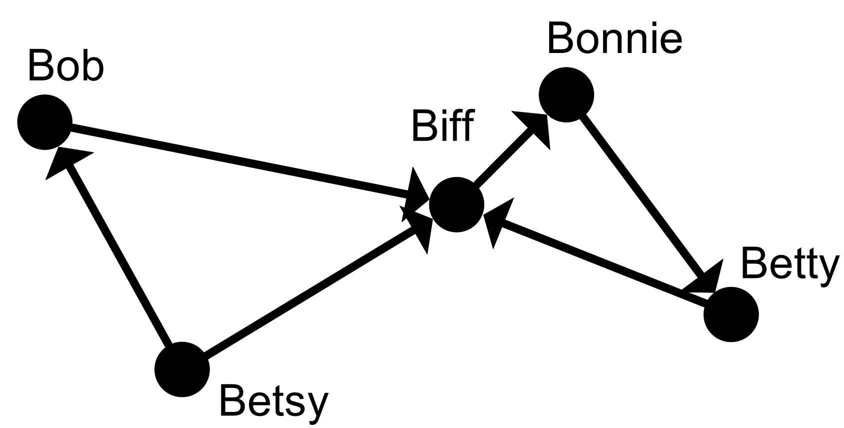


- Ties drawn with arrow heads, which can be in both directions
- The arc set E of a digraph is a binary relation

Directed vs undirected graphs

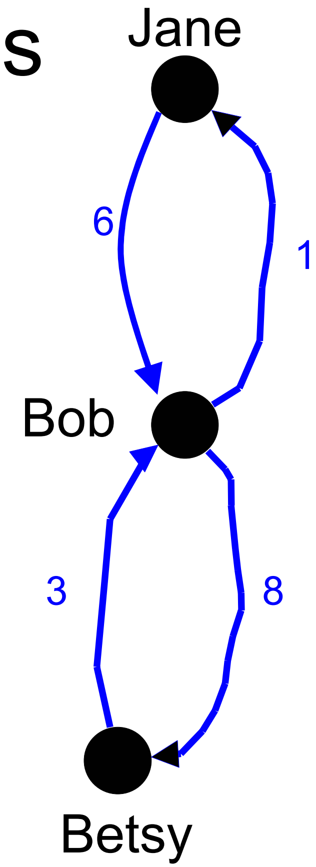
- Undirected relations
  - Attended meeting with
  - Communicates daily with
- Directed relations
  - Lent money to
- Logically vs empirically directed ties
  - Empirically, even un-directed relations can be non-symmetric due to measurement error





Strength of Tie

- We can attach values to ties, representing quantitative attributes
  - Strength of relationship
  - Information capacity of tie
  - Rates of flow or traffic across tie
  - Distances between nodes
  - Probabilities of passing on information
  - Frequency of interaction
- Valued graphs or vigraphs



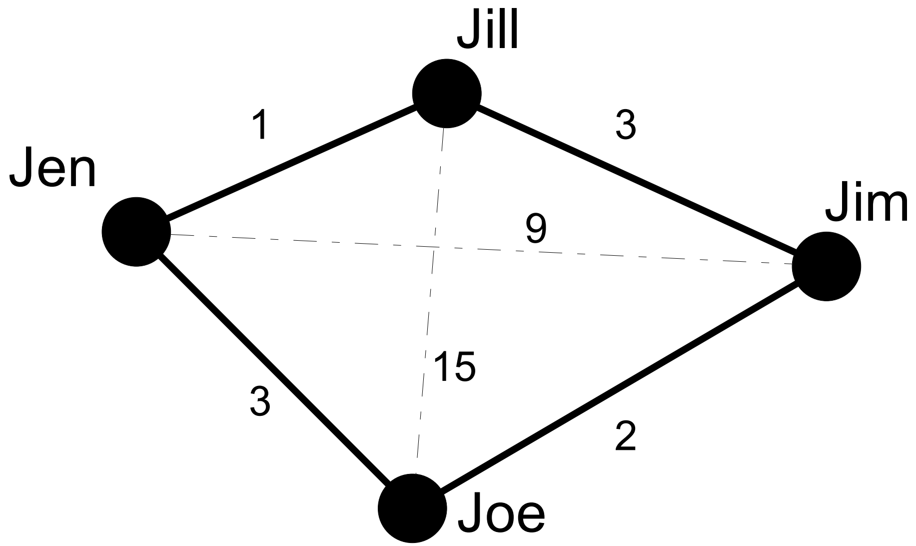
Adjacency Matrices

Friendship

	Jim	Jill	Jen	Joe
Jim	-	1	0	1
Jill	1	-	1	0
Jen	0	1	-	1
Joe	1	0	1	-

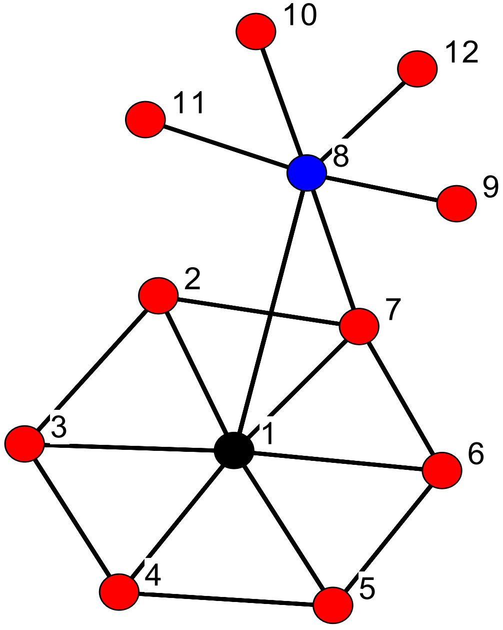
Proximity

	Jim	Jill	Jen	Joe
Jim	-	3	9	2
Jill	3	-	1	15
Jen	9	1	-	3
Joe	2	15	3	-



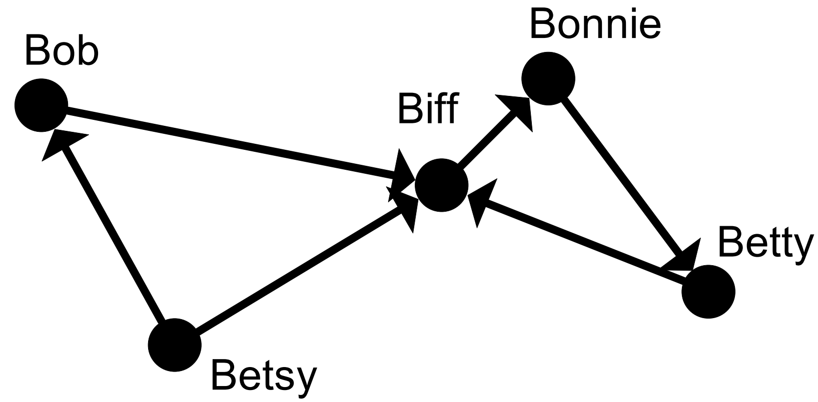
Degree

- Number of edges incident upon a vertex
  - $d_8 = 6$, while $d_{10} = 1$
- Sum of degrees of all nodes is twice the number of edges in graph
- Average degree = density times (n-1)



InDegree & OutDegree
(Directed graphs only)

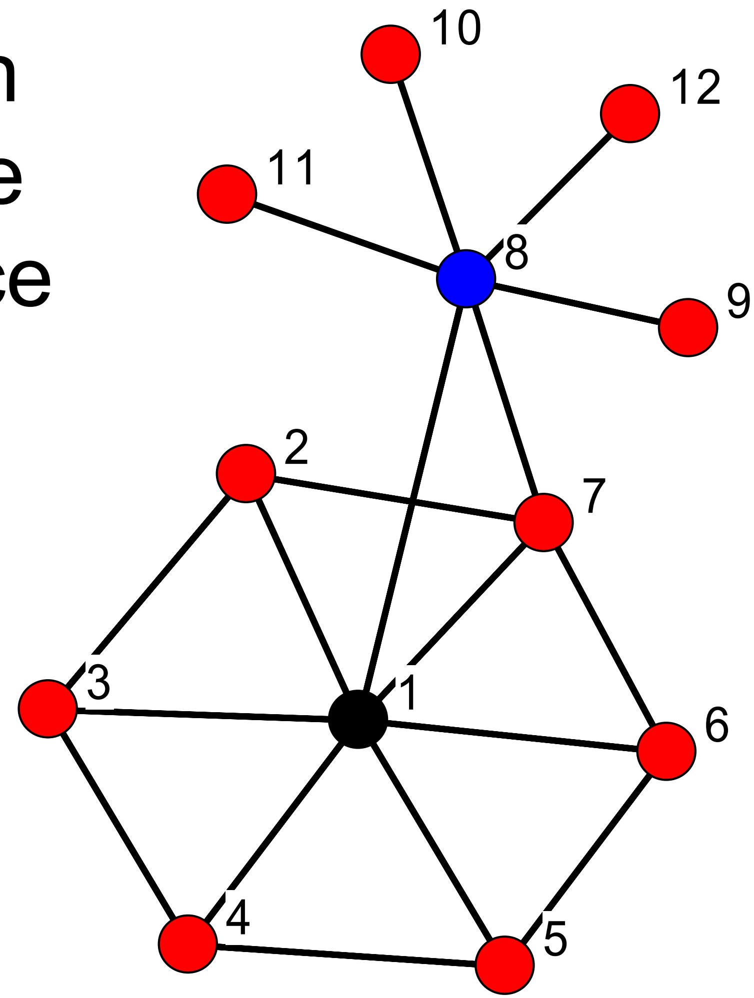
- Indegree is number of arcs that terminate at the node (incoming ties)
  – Indeg(biff) = 3
- Outdegree is number of arcs that originate at the node (outgoing ties)
  – Outdeg(biff) = 1



Average indegree always equals average outdegree

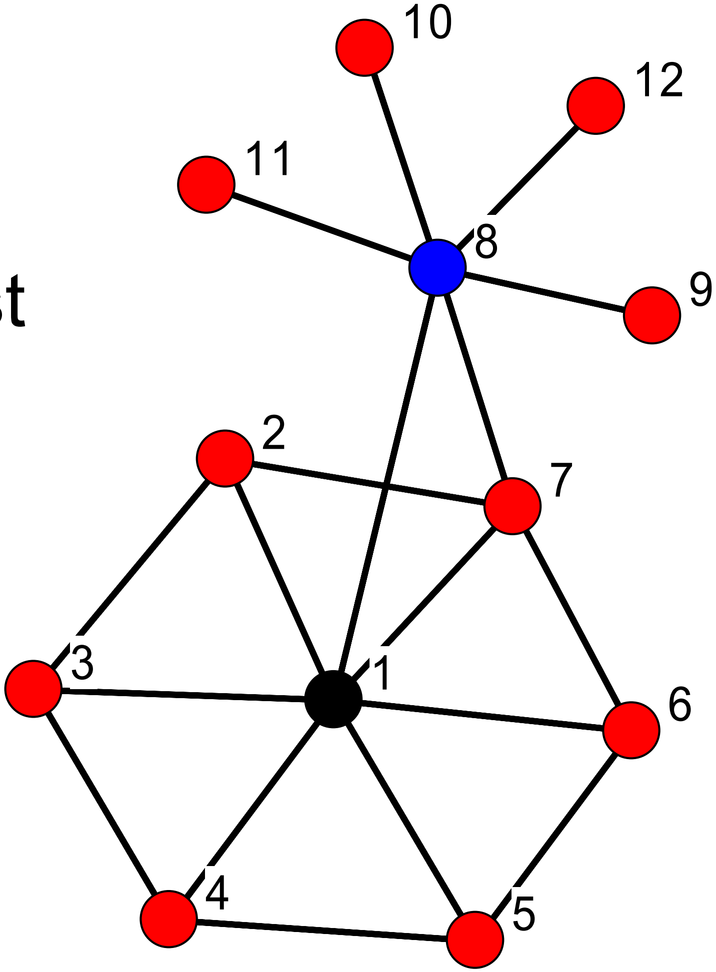
Paths

- Sequence of nodes in which no node can be visited more than once
  - 1-2-3-4-5-6-7-8
  - Not 7-1-2-3-7-4



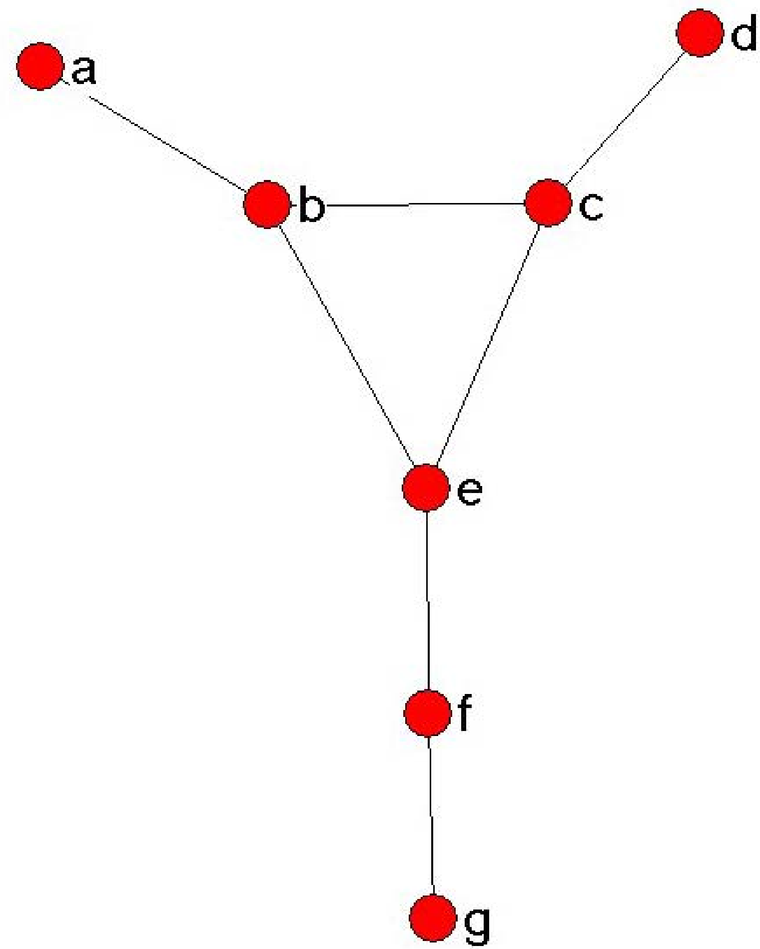
Length & Distance

- Length of a path is number of links
- Distance between two nodes is length of shortest path (aka geodesic)
  - "degrees of separation"



Geodesic Distance Matrix

	a	b	c	d	e	f	g
a	0	1	2	3	2	3	4
b	1	0	1	2	1	2	3
c	2	1	0	1	1	2	3
d	3	2	1	0	2	3	4
e	2	1	1	2	0	1	2
f	3	2	2	3	1	0	1
g	4	3	3	4	2	1	0



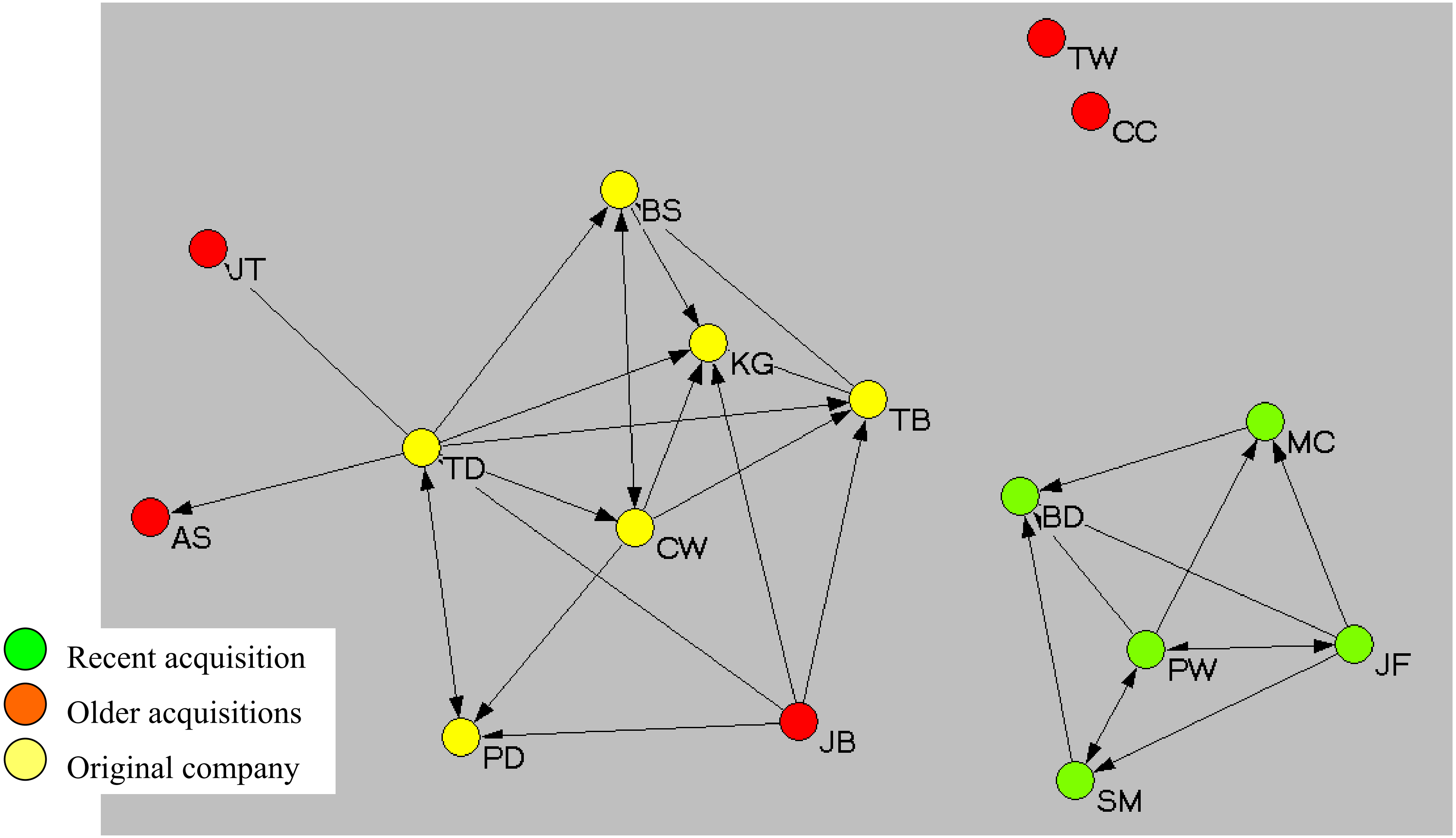
Components

- Maximal sets of nodes in which every node can reach every other by some path (no matter how long)
- A connected graph has just one component

It is relations (types of tie) that define different networks, not components. *A graph that has two components remains one (disconnected) graph.*

A network with 4 components

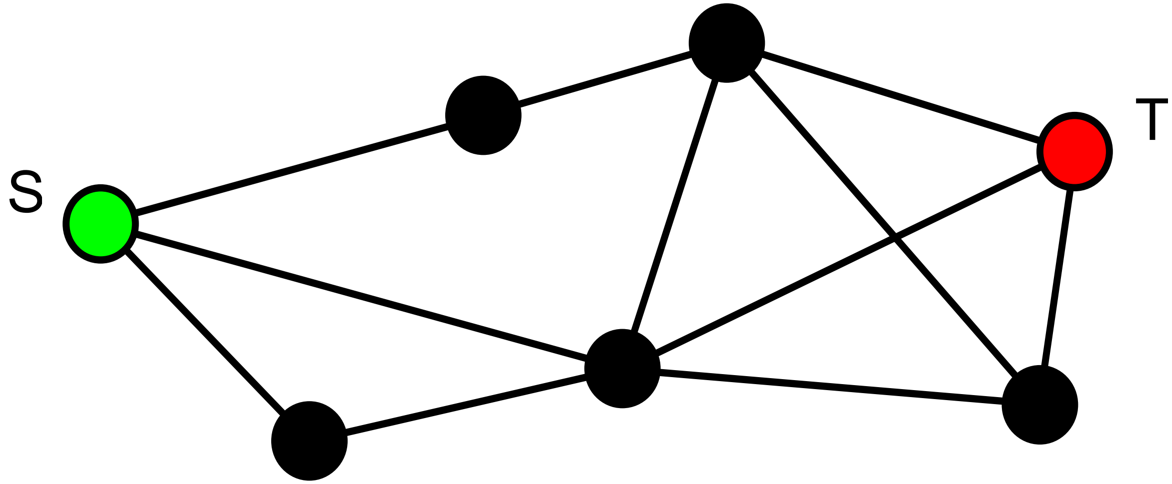
Who you go to so that you can say 'I ran it by ____, and she says ...'



Data drawn from Cross, Borgatti & Parker 2001.

Independent Paths

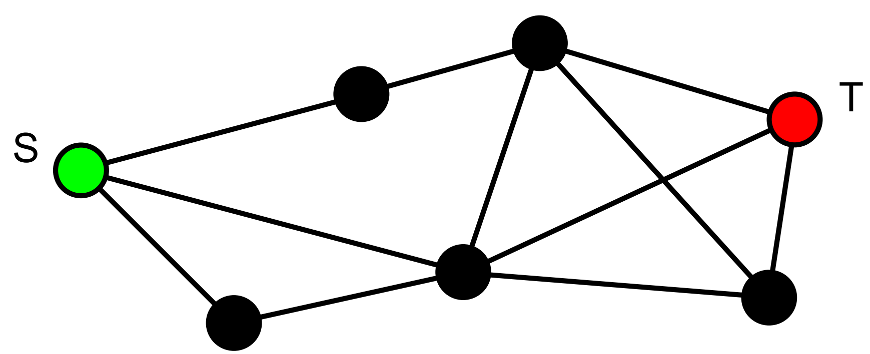
- A set of paths is node-independent if they share no nodes (except beginning and end)
  – They are line-independent if they share no lines



- 2 node-independent paths from S to T
- 3 line-independent paths from S to T

Connectivity

- Line connectivity $\lambda(s,t)$ is the minimum number of lines that must be removed to disconnect s from t
- Node connectivity $\kappa(s,t)$ is minimum number of nodes that must be removed to disconnect s from t

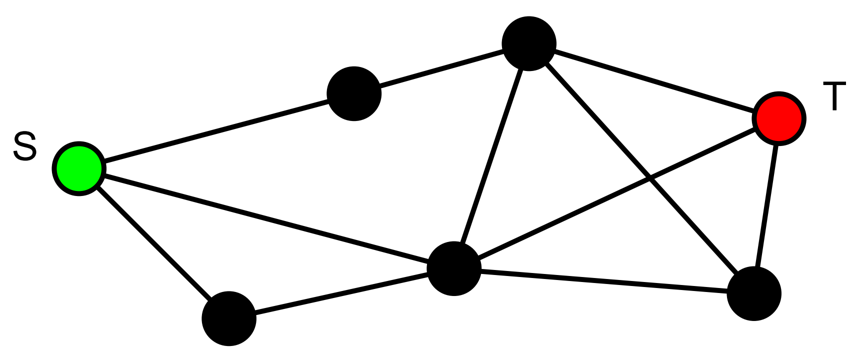


Menger's Theorem

- Menger proved that the number of line independent paths between s and t equals the line connectivity $\lambda(s,t)$
- And the number of node-independent paths between s and t equals the node connectivity $\kappa(u,v)$

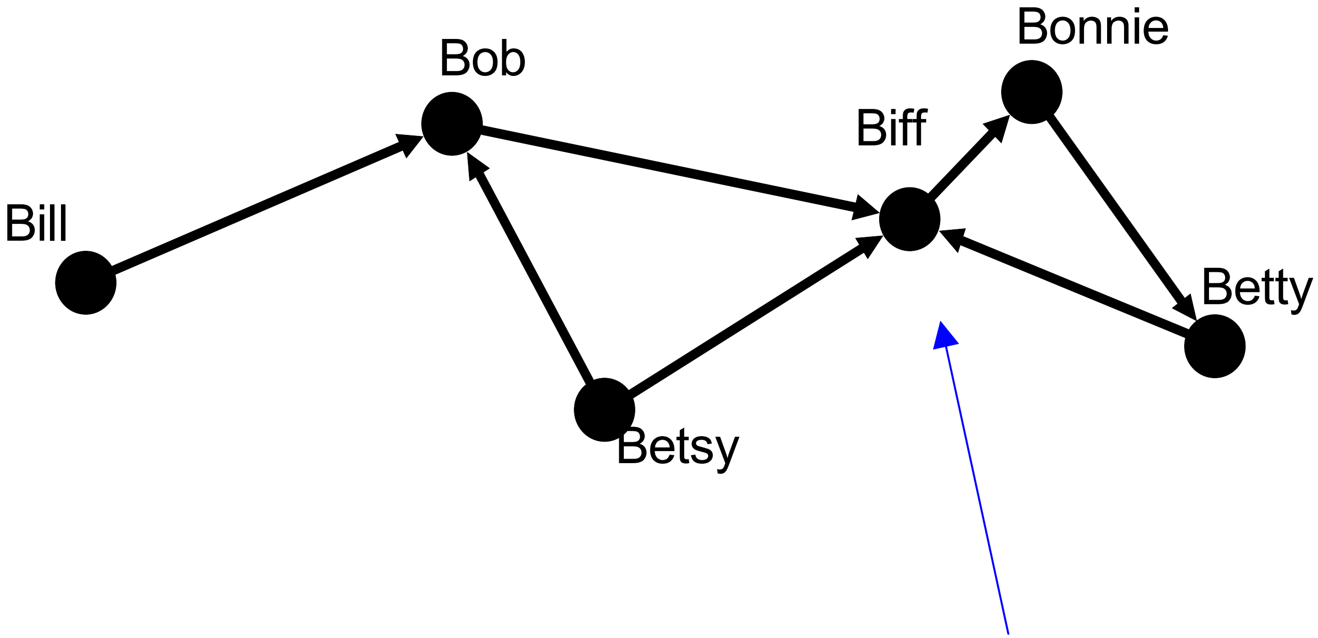
Maximum Flow

- If ties are pipes with capacity of 1 unit of flow, what is the maximum # of units that can flow from s to t?
- Ford & Fulkerson show this was equal to the number of line-independent paths



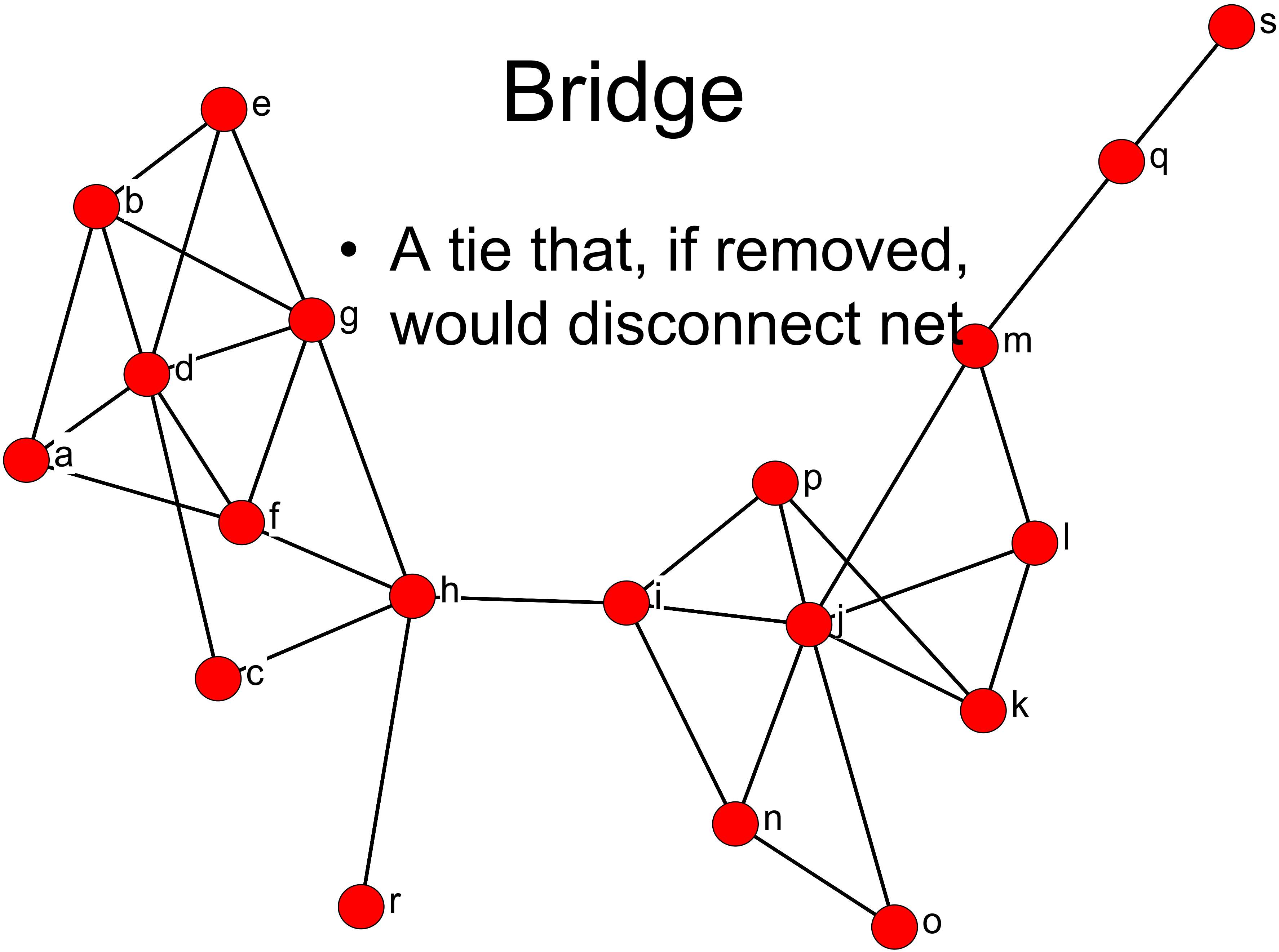
Cutpoints

- Nodes which, if deleted, would disconnect net



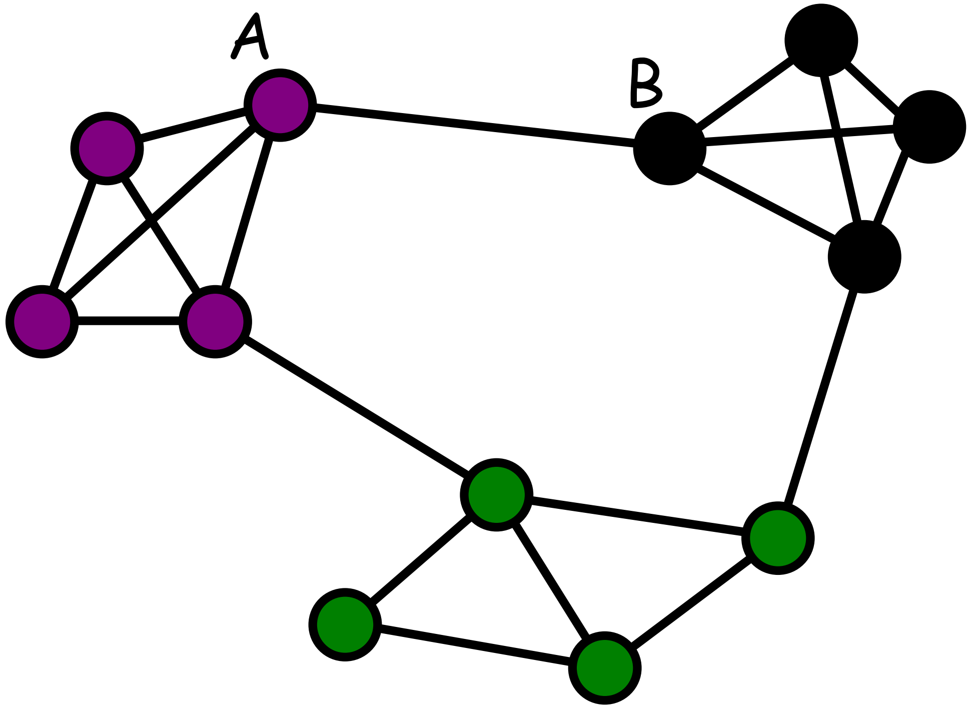
Bridge

- A tie that, if removed, would disconnect net

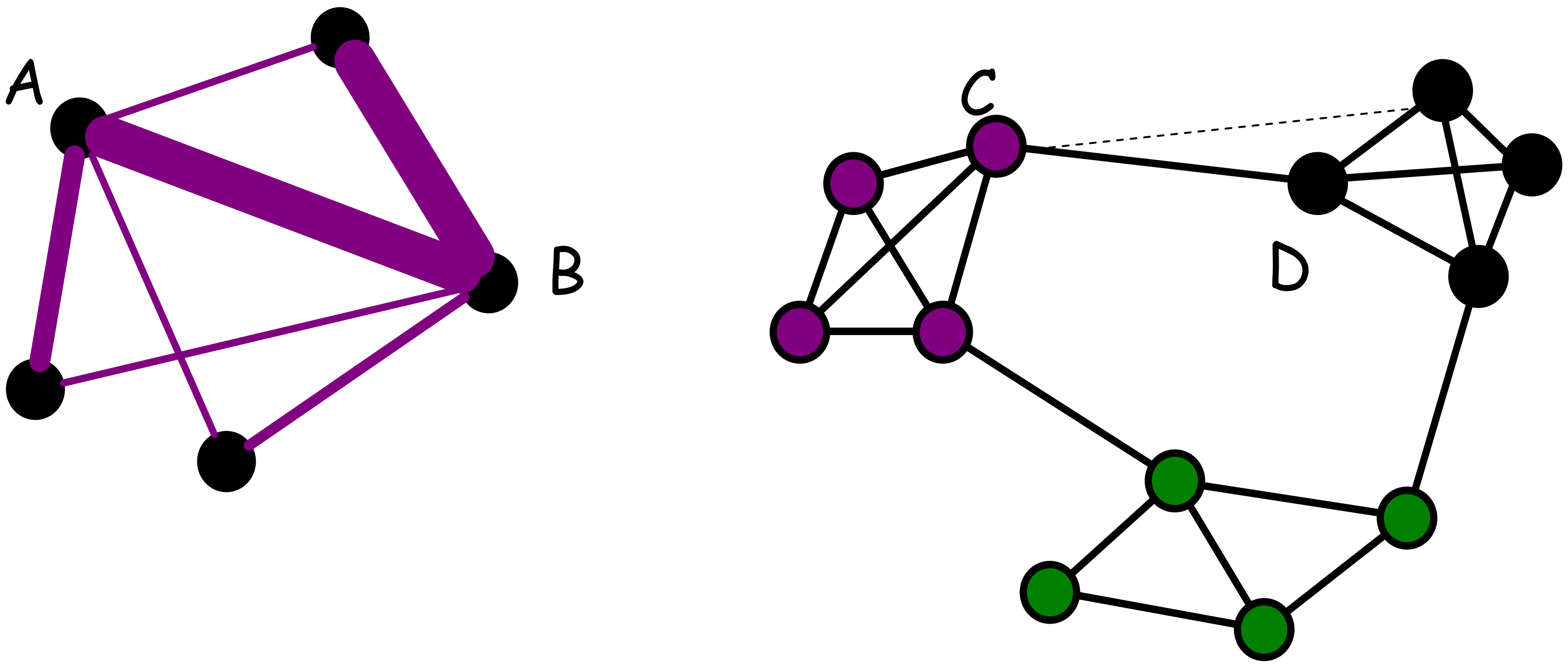


Local Bridge of Degree K

- A tie that connects nodes that would otherwise be at least *k* steps apart



Granovetter Transitivity



Granovetter's SWT Theory

- Strong ties create transitivity
  - Two nodes connected by a strong tie will have mutual acquaintances (ties to same 3rd parties)
- Ties that are part of transitive triples cannot be bridges or local bridges
- Therefore, only weak ties can be bridges
  - Hence the value of weak ties
- Strong ties embedded in tight homophilous clusters, weak ties connect to diversity
  - Weak ties a source of novel information

Walks, Trails, Paths

- Path: can't repeat node
  - 1-2-3-4-5-6-7-8
  - Not 7-1-2-3-7-4
- Trail: can't repeat line
  - 1-2-3-1-7-8
  - Not 7-1-2-7-1-4
- Walk: unrestricted
  - 1-2-3-1-2-7-1-7-1